3E1 Problem Sheet 9

January 12-18, 2004

Lecturer: Claas Röver

1. The *complex conjugate of the complex number* $z = x + iy$, which is usually denoted by $\bar{z}$, is defined by $\bar{z} = x - iy$.

   (a) Let $z_1 = 3 - 2i$ and $z_2 = 4 + i$. In the complex plane, draw the following complex numbers and their complex conjugates: z_1, z_2 and $z_1 + z_2$.

   (b) Show that $\mathrm{Re}(z) = \frac{1}{2}(z + \bar{z})$ and $\mathrm{Im}(z) = \frac{-i}{2}(z - \bar{z})$ hold true for every complex number $z = x + iy$.

   (c) Recall that the *absolute value* (also called *modulus*) of $z = x + iy$, denoted $|z|$, is defined by $|z| = \sqrt{x^2 + y^2}$. Prove the following three identities: $z\bar{z} = |z|^2$, $\overline{z + w} = \bar{z} + \overline{w}$ and $\overline{zw} = \bar{z}\,\overline{w}$.

2. Let $f(z) = \frac{1}{z}$ and write $z = x + iy$.

   (a) Find real functions $u(x, y)$ and $v(x, y)$ such that $f(z) = u(x, y) + iv(x, y)$. (Hint: multiply $f(z)$ by $\frac{\bar{z}}{\bar{z}}$.)

   (b) Show that the functions u and v obtained in part (a) satisfy the Cauchy-Riemann equations.

   (c) Show that $f(z)$ is differentiable everywhere except at 0 and find its derivative.

3. Decide whether there are any points z_0 at which $f(z) = \bar{z}$ is differentiable, where $\bar{z}$ denotes the complex conjugate of z. Justify your answer.