# 1S1 Problem Sheet 6

January 24, 2005, Lecturer: Claas Röver

Solutions to **four** questions are due **Thursday, 3 February** before the lecture. Give your NAME and GROUP NUMBER on the solutions and STAPLE them.

QUESTION 1. (a) Indicate the following complex numbers and their complex conjugates in the complex plane.

$$z_1 = 4 - 2i, \qquad z_2 = 3i, \qquad z_3 = -2 + \tfrac{1}{2}i, \qquad z_4 = 5, \qquad z_5 = \tfrac{1}{\sqrt{2}}(1+i)$$

(b) Using the definitions from part (a), express the complex numbers $z_2$, $z_4$ and $z_5$ in the form $r(\cos\theta + i\sin\theta)$ and find $\theta$ in each case.

QUESTION 2. (a) With the definitions from Question 1(a), express the following complex numbers in the form $a + ib$ with $a, b \in \mathbb{R}$.

$$(i)\ \ z_1 z_2 \qquad (ii)\ \ \tfrac{1}{z_3} \qquad (iii)\ \ \tfrac{z_4}{z_1+z_2} \qquad (iv)\ \ z_5^2 \qquad (v)\ \ \sqrt{z_2}$$

(b) Verify that $|z_1 z_2| = |z_1||z_2|$ for all $z_1, z_2 \in \mathbb{C}$.

QUESTION 3. (a) Verify that $\overline{z_1 z_2} = \overline{z_1}\ \overline{z_2}$ for all $z_1, z_2 \in \mathbb{C}$.

(b) For each of the conditions $(\alpha)$–$(\delta)$ below, sketch the region in the plane whose points, given in polar coordinates $r$ and $\theta$, satisfy

$(\alpha)$ $0 \le r$ and $\theta = \frac{\pi}{3}$ $\qquad$ $(\beta)$ $1 \le r \le 2$ and $\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$

$(\gamma)$ $0 \le r \le 1$ or $\pi \le \theta \le \frac{3\pi}{2}$ $\qquad$ $(\delta)$ conditions $(\alpha)$ and $(\beta)$ above

QUESTION 4. (a) Use polar coordinates to show that the set of all points in the plane whose Cartesian coordinates $x$ and $y$ satisfy $x^2 + y^2 = R^2$ is the circle of radius $|R|$ about the origin, where $R$ is a real constant.

(b) Recall that, in the complex plane, the transition from $z$ to $\overline{z}$ is a reflection in the real axis. Describe the transition from $z$ to $\overline{-iz}\,i$ geometrically and justify your answer.

QUESTION 5. Recall that $0! = 1 = a^0$ for all $a \in \mathbb{R}$ and $\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$ for all integers $n$ and $k$ with $0 \le k \le n$ .

(a) Prove that $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$ for each $n$ and $0 \le k < n$.

(b) Show, by induction on $n$, that for all integers $n \ge 0$

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

*Note.* Part (b) was used to prove $\exp(x+y) = \exp(x)\exp(y)$.