2S1 Problem Sheet 2

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QUESTION 1. Define the gradient of a function $f(x, y)$ of two variables. In your own words, say what the gradient tells about the function f and how it relates to the level curves of f.

QUESTION 2. Determine the gradient of the following functions.

(a) $f(x,y) = x^3y - 2x^2y^2 + 5xy^4$	(b) $f(x,y) = \frac{3xy-x^2}{\sin x}$
(c) $f(x,y) = \sin(2xy^2) + \cos(x - 3y)$	(d) $f(x,y) = \frac{C}{x^2+y^2}$, $C = const.$

QUESTION 3. Let $f(x, y) = 1+\sin(xy-3y)$ and put $(x_0, y_0) = (\frac{10}{3}, \frac{\pi}{2})$. The function f defines a fairly hilly surface in $\mathbb{R}^3$, as plotting it with Mathematica confirms.

(a) Find a formula for the level curve C of f through the point (x_0, y_0) and draw a sketch of it. What is the value f has at every point on C?

(b) Suppose you stand on the surface of f at the point $(x_0, y_0, f(x_0, y_0))$. Find a unit vector in the direction you would choose if you wanted to go downhill as quickly as possible.

(c) Find the directional derivative of f in the direction of the vector $(\cos t, \sin t)$. (Hint: you should get a function of the three variables x, y and t.)

QUESTION 4. (a) Find the directional derivative of $f(x, y) = \frac{5}{x^2+y^2}$ at the point P towards the origin when

(i) $P = (2, 1)$ (ii) $P = (-1, 2)$ (iii) $P = (\sqrt{5}, 0)$.

(b) Find the directional derivative of $f(x, y) = Ax^2 + Bxy + Cy^2$ at (a, b) in the direction from (a, b) towards (b, a), where A, B and C are constants.

(c) Give a reason why you always got the same answer in part (a) above.

QUESTION 5. Let $f(x, y)$ be a function of the independent variables x and y with continuous second-order partial derivatives. Show that, if $x = e^{\frac{1}{2}(u-v)}$ and $y = e^{\frac{1}{2}(u+v)}$, then

$$\frac{\partial^2 f}{\partial u^2} - \frac{\partial^2 f}{\partial v^2} = axy\frac{\partial^2 f}{\partial x \partial y}$$

and determine the constant a.

QUESTION 6. Find the extreme points, that is relative maxima and minima and saddle points, of the following functions of two variables.

(a) $f(x,y) = 2x^2 + y^2 - xy + 7y$	(b) $f(x,y) = -xye^{-\frac{1}{2}(x^2+y^2)}$
(c) $f(x,y) = y^2 - xy + 2x + y + 1$	(d) $f(x,y) = x \sin y$