2S1 Problem Sheet 5

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QUESTION 1. (a) Let R be the rectangle in the uv-plane which is bounded by the lines $u = 1$, $u = 2$, $v = 0$ and $v = \frac{\pi}{2}$. Find the image of R in the xy-plane under the transformation $T(u, v) = (u \cos v, u \sin v)$ and compute the Jacobian of T.

(b) Use an appropiate change of variables to evaluate $\iint_R (x - y)e^{x^2-y^2} dA$, where R is the region bounded by the lines $x + y = 0$, $x + y = 1$, $x - y = 1$ and $x - y = 4$.

QUESTION 2. Let S be surface given by $x(u, v) = \sqrt{4 - v} \cos u$, $y(u, v) = \sqrt{4 - v} \sin u$ and $z(u, v) = v$ with $0 \le u \le 2\pi$ and $0 \le v \le 4$.

(a) Describe the constant u- and v-curves of S.

(b) Compute the surface area of S.

(c) Find a function $f(x, y)$ and a region R in the xy-plane so that the graph/surface of f over the region R coincides with S. Justify your answer.

(d) Determine the volume that is enclosed by S and the xy-plane.

QUESTION 3. (a) Let $C(t) = (2t, t^2 - 1)$, $1 \le t \le 2$. Sketch the path C indicating its direction and find its length.

(b) Evaluate the line integrals $\int_C 2(x - y)(1 + \cos x)\mathrm{d}s$ and $\int_C y\mathrm{d}x + (y - x)\mathrm{d}y$, where C is the path from $(0, 0)$ to $(2\pi, 2\pi)$ with $x(t) = t$ and $y(t) = t + \sin t$.

QUESTION 4. Let $\mathbf{F}$ and $\mathbf{G}$ be 3-dimensional vector fields.

(a) Show that $\operatorname{div}(\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot \operatorname{curl} \mathbf{F} - \mathbf{F} \cdot \operatorname{curl} \mathbf{G}$

(b) Give a geometric argument for the fact that $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for any 3 dimensional vectors $\mathbf{a}$ and $\mathbf{b}$.

(c) Show that, if the component functions of the vector field $\mathbf{F}$ have continuous second partial derivatives, then $\operatorname{div}(\operatorname{curl} \mathbf{F}) = 0$.
Remark. The latter equation can also be written as $\nabla \cdot (\nabla \times \mathbf{F})$ which is similar to the result in part (*b*).

QUESTION 5. (a) Show that the vector field $\mathbf{F}(x, y, z) = (6xy - \sin z, 3x^2, -x \cos z)$ is conservative, by finding a potential function. Then find the work done by $\mathbf{F}$ on a particle moving from the origin to the point $(1, 1, \pi)$.

(b) Let $\mathbf{F} = (x - y, z - x, \frac{8}{3}y)$. Compute the work done by $\mathbf{F}$ on a particle moving from the origin to the point $(1, 1, 1)$ first a long a straight line and then along the path $C(t) = (t, t^2, t^3)$, $0 \le t \le 1$. Decide whether $\mathbf{F}$ is conservative.