CS3304 Logic – Problem Sheet 3

October 7, 2016, Lecturer: Claas Röver

QUESTION 1. Give derivations of the following using the natural deduction rules other than reduction ad absurdum.

(a) $\vdash A \to (B \to (A \wedge B))$

(b) $\{A \to B, B \to C\} \vdash A \to C$

(c) $\{A \to B\} \vdash \neg(A \wedge \neg B)$

(d) $\vdash (A \to (B \wedge C)) \to ((A \to B) \wedge (A \to C))$

QUESTION 2. Write out derivations of the following using the natural deduction rules.

(a) $\vdash (\neg(\neg A)) \to A$

(b) $\{(\neg A) \to (\neg B)\} \vdash B \to A$

(c) $\vdash A \to ((\neg A) \to B)$

QUESTION 3. Recall the definition of well formed formulæ (or **wff** for short) over the signature with variables $\{p_1, p_2, p_3, \ldots\}$ and special symbols $\{(,), \neg, \to\}$

- Every variable is a **wff**.
- If ϕ is a **wff**, then so is $(\neg\phi)$,
- If ϕ and ψ are **wff**, then so is $(\phi \to \psi)$.

Decide for each of the following strings, whether it is a **wff**.

i.	$((\neg p_2)(\neg p_4))$	iii.	$(\neg(\neg(p_1)))$	v.	$p_1 \to p_1$
ii.	$\neg(p_3 \to p_2)$	iv.	$((p_{10} \to p_1) \to \neg p_1)$	vi.	$((p_1 \to p_2) \to (\neg(\neg(p_{82} \to p_2))))$

QUESTION 4. You are allowed to use the rules

$$S \rightsquigarrow (\neg S), \qquad S \rightsquigarrow (S \to S) \quad \text{and} \quad S \rightsquigarrow p_i \text{ for } i \geq 1$$

to generate strings over the alphabet $\{S, (,), \neg, \to, p_{1,2}, \ldots\}$, starting from the string S, and in each step replacing one occurrence of S by the right-hand side of one of the rules. For example

$$S \rightsquigarrow (S \to S) \rightsquigarrow ((\neg S) \to S) \rightsquigarrow ((\neg p_2) \to S)$$

is a valid generation of a string.

(a) Using induction on the number of applications of rules, prove that if σ is a string that does not contain S and can be generated in his way, then σ is a **wff**, as defined in Question 3.

(b) Prove also the converse. Namely, that every **wff** can be generated from S using the rules. *Hint:* This is also an induction. This time on a number assigned to a **wff**that you have to define first.

(c) Prove that for every **wff** ϕ there is a left-most derivation of ϕ form S, i.e. a derivation in which every application of a rule replaces the left-most occurrence of S.