Natural Deduction Rules

Below is the list of natural deduction rules invented by G. Gentzen in 1935.

<u>$\wedge$-introduction:</u> $\dfrac{A \quad B}{A \wedge B}$

<u>$\wedge$-elimination:</u> $\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}$

<u>$\vee$-introduction:</u> $\dfrac{A}{A \vee B} \qquad \dfrac{B}{A \vee B}$

<u>$\vee$-elimination:</u> $\dfrac{A \vee B \quad \begin{array}{c}\cancel{A}\\ \vdots \\ C\end{array} \quad \begin{array}{c}\cancel{B}\\ \vdots \\ C\end{array}}{C}$

<u>$\to$ -introduction:</u> $\dfrac{\begin{array}{c}\cancel{A}\\ \vdots \\ C\end{array}}{A \to C}$

<u>modus ponens:</u> $\dfrac{A \qquad A \to C}{C}$

<u>absurdity:</u> $\dfrac{\bot}{A}$

<u>identity:</u> $\dfrac{A}{A}$

<u>reductio ad absurdum:</u> $\dfrac{\begin{array}{c}\cancel{\neg A}\\ \vdots \\ \bot\end{array}}{A}$

These rules allow us to deduce conclusions from a set of assumptions. We use the notation $\{A_1, A_2, \ldots, A_n\} \vdash C$ to mean that the conclusion C can be deduced from the assumptions A_i for $1 \leq i \leq n$ using only the above rules. Here is an example proving

$$\{(A \wedge B) \to C\} \vdash A \to (B \to C).$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\cancel{A}^{(2)} \qquad \cancel{B}^{(1)}}{A \wedge B} \qquad (A \wedge B) \to C}{C}}{B \to C}}{A \to (B \to C)}$$

simply make these assumptions, discharge below
$\wedge$-introduction + given assumption
modus ponens
$\to$-introduction, discharging B (1)
$\to$-introduction, discharging A (2)

This is a proof because we arrived at the desired conlusion $(A \to (B \to C))$ and have only one undischarged assumption left in the second row.