Predicate Logic aka First-Order Logic

Predicate logic is a formal system consisting of the following four ingredients.

An alphabet $\Sigma = \{(,),,,\neg,\rightarrow,\forall,\exists\}\dot\cup\mathcal{P}\dot\cup\mathcal{F}\dot\cup\mathcal{C}\dot\cup V$ including a set $\mathcal{P}$ of *predicate letters* P_i^n, a set $\mathcal{F}$ of *function letters* f_i^n, a set $\mathcal{C}$ of *constant symbols* c_i and a set V of *variables* x_i, $i,n \geq 0$.

Terms which are defined recursively:

- Every variable $x_i \in V$ and every constant $c_i \in \mathcal{C}$ is a term.
- If $t_1,\ldots,t_n$ are terms, then so is $f_i^n(t_1,\ldots,t_n)$ for every n-ary function letter $f_i^n \in \mathcal{F}$.
- Nothing else is a term.

Well formed formulæ (**wff** for short) are also defined recursively:

- $P_i^n(t_1,\ldots,t_n)$ is a **wff** for any terms $t_1,\ldots,t_n$ and n-ary predicate letter $P_i^n \in \mathcal{P}$.
- If A and B are **wff**, then so are $(\neg A)$ and $(A \rightarrow B)$.
- If A is a **wff**, then so are $(\forall x_i)A$ and $(\exists x_i)A$ for any variable $x_i \in V$.
- Nothing else is a **wff**.

Five axiom schemes for any **wff** A, B and C:

Ax1: $(A \rightarrow (B \rightarrow A))$
Ax2: $((A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C)))$
Ax3: $(((\neg B) \rightarrow (\neg A)) \rightarrow (A \rightarrow B))$
Ax4: $(\forall x_i)A(x_i) \rightarrow A(t),$ where the term t is free for x_i in A
Ax5: $(\forall x_i)(A \rightarrow B) \rightarrow (A \rightarrow (\forall x_i)B),$ if there is no free x_i in A

Deductions are sequences of **wff** in which every entry is either an (instance of an) axiom, a hypothesis or obtained from previous entries in the sequence $\mathcal{D}$ using either

Modus Ponens (MP): If $A, (A \rightarrow B) \in \mathcal{D}$, then $\mathcal{D} \cup \{B\}$ is a deduction, for any $A, B \in \mathcal{W}$; or
Generalisation (G): If $A \in \mathcal{D}$, then $\mathcal{D} \cup \{(\forall x_i)A\}$ is a deduction, for any $A \in \mathcal{W}$ and $x_i \in V$.

If C can be deduced from the hypotheses $A_1, A_2, \ldots A_n$ we write $\{A_1, A_2, \ldots A_n\} \vdash C$.

Deduction Theorem (DT). For $\Delta \subseteq \mathcal{W}$ and $A \in \mathcal{W}$, if $\Delta \cup \{A\} \vdash B$ using (G) only with x_i which is not free in A, then $\Delta \vdash A \rightarrow B$.

Proof. Same as in APL plus the case when B was deduced using (G), i.e. $B = (\forall x_i)C$ and x_i is not free in A. Here, $\mathcal{IH}$ gives $\Delta \vdash A \rightarrow C$, whence $\Delta \vdash (\forall x_i)(A \rightarrow C)$ by (G), and so $A \rightarrow (\forall x_i)C$ from Ax5 and MP.

Transitivity of Implication (TI). $\{A \rightarrow B,\ B \rightarrow C\} \vdash A \rightarrow C$ (As in APL from DT; no (G) needed.)

Interpretations give rise to semantics/meaning. An interpretation $\mathcal{I}$ consists of a non-empty domain $D_\mathcal{I}$ of objects, assignments of objects to the constant symbols $\mathcal{I}(c_i) \in D_\mathcal{I}$, as well as choices of n-ary relations $\mathcal{I}(P_i^n) \subseteq {D_\mathcal{I}}^n$ and an n-ary functions $\mathcal{I}(f_i^n)\colon D_\mathcal{I}^n \rightarrow D_\mathcal{I}$ for all symbols P_i^n and f_i^n.

A **valuation** v in an interpretation $\mathcal{I}$ assigns objects in $D_\mathcal{I}$ to all the variables, and hence to all terms, by setting $v(c_i) = \mathcal{I}(c_i)$ and (recursively) defining $v(f_i^n(t_1,\ldots,t_n)) = \mathcal{I}(f_i^n)(v(t_1),\ldots,v(t_n))$. We write $\mathcal{I} \models_v A$ if the valuation v in the interpretation $\mathcal{I}$ *satisfies* the **wff** A, which is defined recursively:

- $\mathcal{I} \models_v P_i^n(t_1,\ldots,t_n)$ if and only if $(v(t_1),\ldots,v(t_n)) \in \mathcal{I}(P_i^n)$;
- $\mathcal{I} \models_v (\neg A)$ if and only if $\mathcal{I} \not\models_v A$, i.e. $\mathcal{I} \models_v A$ does not hold;
- $\mathcal{I} \models_v (A \rightarrow B)$ if and only if $\mathcal{I} \models_v B$ or $\mathcal{I} \models_v \neg A$;
- $\mathcal{I} \models_v (\forall x_i)A$ if and only if $\mathcal{I} \models_u A$ for all valuations u in $\mathcal{I}$ with $v(x_j) = u(x_j)$ for $j \neq i$.
- $\mathcal{I} \models_v (\exists x_i)A$ if and only if $\mathcal{I} \models_u A$ for some valuation u in $\mathcal{I}$ with $v(x_j) = u(x_j)$ for $j \neq i$.

Lemma. If v is a valuation in an interpretation $\mathcal{I}$, then $\mathcal{I} \models_v (\exists x_i)A$ if and only if $\mathcal{I} \models_v \neg(\forall x_i)(\neg A)$.

The **wff** A is *valid* or *true in the interpretation* $\mathcal{I}$, written $\mathcal{I} \models A$, if $\mathcal{I} \models_v A$ for all valuations v in $\mathcal{I}$. And A is called *logically valid* or simply *valid*, written $\models A$, if $\mathcal{I} \models A$ for all interpretations $\mathcal{I}$.

Theorem. Predicate Logic is *sound* ($\vdash A$ implies $\models A$), *consistent* (not both $\vdash A$ and $\vdash \neg A$) and *complete* ($\models A$ implies $\vdash A$).

Proof of soundness. As for APL, this is by induction on the length of a deduction sequence for A:

Induction Base: $\not\models$ Ax1 $\Rightarrow \models A$ & $\not\models (B \rightarrow A)$ $\Rightarrow \models B$ & $\not\models A$, contradiction; Ax2 and Ax3 similarly.
$\not\models$ Ax4 $\Rightarrow \models (\forall x_i)A$ & $\not\models A(t)$, contradiction because t is free for x_i in A.
$\not\models$ Ax5 $\Rightarrow \models (\forall x_i)(A \rightarrow B)$ & $\models A$ & $\not\models (\forall x_i)B$ $\Rightarrow$ there are $\mathcal{I}$ and v with $\mathcal{I} \not\models_v B$ $\Rightarrow \mathcal{I} \not\models_v (A \rightarrow B)$ (as $\models A$) $\Rightarrow \not\models (\forall x_i)(A \rightarrow B)$, contradiction

Induction Step: $\models A$ & $\models (A \rightarrow B)$ $\Rightarrow \models A$ & ($\not\models A$ or $\models B$) $\Rightarrow \models B$, so MP is sound.
$\models A \Rightarrow \models (\forall x_i)A$, because this depends on less values than $\models A$, so G is sound.

Completeness is known as Gödel's Completeness Theroem and not proved here.