Semantic Tableaux

Semantic tableaux are rooted trees whose nodes are labelled by sets of **wff**.

In propositional logic, the rules for extending such trees all follow from the three basic rules:

$$
\begin{array}{c} \neg\neg A \\ | \\ A \\ \end{array}
\qquad
\begin{array}{c} A \wedge B \\ | \\ A \\ B \end{array}
\qquad
\begin{array}{cc} \multicolumn{2}{c}{A \vee B} \\ / & \backslash \\ A & B \\ & \end{array}
$$

The derived rules are:

$$
\begin{array}{c} \neg(A \vee B) \\ | \\ \neg A \\ \neg B \end{array}
\quad
\begin{array}{c} \neg(A \to B) \\ | \\ A \\ \neg B \end{array}
\quad
\begin{array}{cc} \multicolumn{2}{c}{\neg(A \wedge B)} \\ / & \backslash \\ \neg A & \neg B \\ & \end{array}
\quad
\begin{array}{cc} \multicolumn{2}{c}{A \to B} \\ / & \backslash \\ \neg A & B \\ & \end{array}
\quad
\begin{array}{cc} \multicolumn{2}{c}{A \leftrightarrow B} \\ / & \backslash \\ A & \neg A \\ B & \neg B \end{array}
\quad
\begin{array}{cc} \multicolumn{2}{c}{\neg(A \leftrightarrow B)} \\ / & \backslash \\ A & \neg A \\ \neg B & B \end{array}
$$

These rules are such that every finite set of **wff** of propositional logic is eventually split up into atomic constituents. **Heuristic:** Apply non-branching rules before branching rules.

In first order logic, there are the following additional rules:

$$
\begin{array}{c} \neg(\forall x)A(x) \\ | \\ (\exists x)\neg A(x) \end{array}
\quad
\begin{array}{c} \neg(\exists x)A(x) \\ | \\ (\forall x)\neg A(x) \end{array}
\quad
\begin{array}{c} (\forall x)A(x) \\ | \\ A(t) \text{ for any term } t \end{array}
\quad
\begin{array}{c} (\exists x)A(x) \\ | \\ A(t) \text{ for any term } t \textbf{ not used} \\ \text{in the tableau so far} \end{array}
$$

The last two are *instantiation rules*. **Heuristic:** Always do $\exists$-instantiation before $\forall$-instantiation.

A branch of a tableau is *closed* if it contains a formula ϕ and its negation $\neg\phi$. The tableau is *closed* if all its branches are closed. NOTE: $\forall$-formulæ can be instantiated for every possible term and hence potentially infinitely many times, so that tableaux can have infinite branches.

Main Fact. A set S of **wff** is inconsistent if and only if there is a closed semantic tableaux for S. WARNING: This does not mean that you can conclude that S is consistent, if a certain tableaux for S does not close. For that you have to argue that every possible tableau for S does not close! Here are two examples.

Proving $(\exists x)(\exists y)R(x,y) \to (\exists y)(\exists x)R(x,y)$

$$
\begin{array}{c}
\neg((\exists x)(\exists y)R(x,y) \to (\exists y)(\exists x)R(x,y)) \\
| \\
(\exists x)(\exists y)R(x,y) \\
\neg(\exists y)(\exists x)R(x,y) \\
| \\
(\forall y)\neg(\exists x)R(x,y) \\
| \\
(\forall y)(\forall x)\neg R(x,y) \\
| \\
(\exists y)R(c,y) \\
| \\
R(c,d) \\
| \\
(\forall x)\neg R(x,d) \\
| \\
\neg R(c,d) \\
\hline
\text{closed}
\end{array}
$$

The following is a valid argument:
If something eats mice, then it is a cat or a bird of prey. Birds of prey are not mammals, but cats are. This beast eats mice and is a mammal. Therefore it must be a cat.
Let $E(x)$, $C(x)$, $B(x)$ and $M(x)$ mean x **e**ats mice, is a **c**at, is a **b**ird of prey and is a **m**ammal, respectively.

$$
\begin{array}{c}
(\forall x)(E(x) \to (C(x) \vee B(x))) \\
(\forall x)((B(x) \to \neg M(x)) \wedge (C(x) \to M(x))) \\
E(b) \wedge M(b) \quad (b \text{ is ‘this beast’}) \\
\neg C(b) \quad \text{(negated conclusion)} \\
| \\
E(b) \\
M(b) \\
| \\
E(b) \to (C(b) \vee B(b)) \\
\begin{array}{cc}
/ & \backslash \\
\begin{array}{c} \neg E(b) \\ \hline \text{closed} \end{array} &
\begin{array}{c}
C(b) \vee B(b) \\
\begin{array}{cc}
/ & \backslash \\
B(b) & \begin{array}{c} C(b) \\ \hline \text{closed} \end{array}
\end{array}
\end{array}
\end{array} \\
| \\
(B(b) \to \neg M(b)) \wedge (C(b) \to M(b)) \\
| \\
B(b) \to \neg M(b) \\
C(b) \to M(b) \\
\begin{array}{cc}
/ & \backslash \\
\begin{array}{c} \neg B(b) \\ \hline \text{closed} \end{array} &
\begin{array}{c} \neg M(b) \\ \hline \text{closed} \end{array}
\end{array}
\end{array}
$$