MA203 Linear Algebra – Problem Sheet 1

January 20, 2017, Lecturer: Claas Röver

Question 1. Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. First state the definition of the norm of v, written $\|v\|$. Then prove that $\|\alpha v\| = |\alpha|\|v\|$ for any $\alpha \in \mathbb{R}$, where $|\alpha|$ denotes the absolute value of α. *Hint:* A direct calculation.
Now deduce that $\left\| \dfrac{v}{\|v\|} \right\| = 1$.

Question 2. Let $u, v, w \in \mathbb{R}^n$. First state the definition of the dot product $v \cdot w$. Then prove that $u \cdot (v + w) = u \cdot v + u \cdot w$. *Hint:* Another direct calculation.

Question 3. For each of the following expressions decide whether it makes sense, and if so evaluate it.

(a) $4(2, 3) - (3, 1, 0)$	(d) $5 \cdot (-2, 6)$	(g) $(2, 0, 0) \cdot ((-1, 1, 0) + (1, 0, 1))$
(b) $(2, -2) \cdot (-3, 5)$	(e) $(4, 2, 1)/4$	(h) $3((\frac{2}{3}, \frac{1}{3}) + \frac{1}{3}(-2, -1))$
(c) $(1, 0) \cdot (1, 1, 1)$	(f) $(5, 2)/(6, 3)$	(k) $7\|(-1, 2, 1, -2)\|$

Question 4. Consider the following vectors in $\mathbb{R}^2$:

$$p = (2,\ -1), \quad q = (3,\ 2.5), \quad r = (-3,\ 4) \text{ and } s = (-1,\ 1).$$

(a) Calculate $p - q$ and $p + q$ and then draw them in a sketch together with p and q.

(b) Calculate the norms of r and s.

(c) Calculate the dot products $\alpha = p \cdot q$ and $\beta = r \cdot q$. In a diagram draw the five vectors

$$u = \frac{\alpha}{\|p\|^2} p, \quad v = \frac{\beta}{\|r\|^2} r, \quad p, \quad r \text{ and } u.$$

Describe the vector u in relation to p and q in words.

(d) Give a parametric description of the line through p and r, and find the points where the line intersects the coordinate axes.

Question 5. Calculate the point, if it exists, where the line through the origin and $(2, 3, 5)$ intersects the plane through the three points $(1, -1, 2)$, $(0, -2, 1)$ and $(2, 3, -1)$.

Question 6. Let $u, v, w \in \mathbb{R}^n$ and assume that v and w are non-zero and perpendicular to each other. Show that $\frac{u \cdot v}{\|v\|^2} v$ is the component of u in the direction of v.

Question 7. (a) Given $p = (2.2, -0.3) \in \mathbb{R}^2$, find all vectors in $\mathbb{R}^2$ that are perpendicular to p and have norm equal to 1.

(b) Given $p = (p_1, p_2) \in \mathbb{R}^2$, find all vectors in $\mathbb{R}^2$ that are perpendicular to p and have norm equal to 1.

(c) How many solutions does the problem in part (b) have, when $p \in \mathbb{R}^3$? What if $p \in \mathbb{R}^4$? Do these solutions form particular shapes, and if so which?

Question 8. Is it true that, if the n-dimensional vectors p and q are perpendicular to each other, then $p + q$ is perpendicular to $p - q$? If not, can you give a condition that makes it true?

Question 9. In your own words explain the following words used frequently in mathematics: deduce, calculate, prove, decide, justify, verify, give an example, state, describe, evaluate.