MA211 Calculus I – Problem Sheet 1

September 12, 2016, Lecturer: Claas Röver

QUESTION 1. For each of the following functions, determine its domain and range and decide whether it is one-to-one.

(a) $f(x) = x^3$ (b) $f(x) = \dfrac{1}{3x-5}$ (c) $f(x) = x^2 - 2$

(d) $g(x) = \sqrt{2-x}$ (e) $g(t) = e^t$ (f) $g(x) = (x-2)^2$

QUESTION 2. For each of the following functions, give a largest domain on which it is one-to-one and then invert the function on that domain.

(a) $f(x) = \dfrac{1}{(x+2)^2}$ (b) $f(x) = \dfrac{1}{3x-5}$ (c) $f(x) = x^2 - 2x$

(d) $g(x) = \sqrt{2-x}$ (e) $g(t) = e^t$ (f) $g(x) = (x+2)^2$

QUESTION 3. For each of the following statements decide whether it is true or false.

(a) Every function is invertible.
(b) A function that is invertible must be continuous.
(c) A function is injective if and only if it is one-to-one.
(d) Every one-to-one function has an inverse.
(e) A function that has an inverse must be injective.
(f) A function f is injective if $x_1 = x_2$ implies $f(x_1) = f(x_2)$.
(g) If the function f is invertible, then there exists a function g such that $g(f(x)) = x$ for all x in the domain of f.

QUESTION 4. Give the definition of an injective function. Then argue that a function f is injective, if its derivative f' is continuous and never zero.
Hint: Recall the notion of a monotonic funtion.

QUESTION 5. Show that $f(x) = \dfrac{4x^3}{x^2+1}$ has an inverse and find $(f^{-1})'(2)$.

QUESTION 6. Assume that the function $f(x)$ satisfies $f'(x) = \dfrac{1}{x}$, and that f is one-to-one. If $y = f^{-1}(x)$, show that $\dfrac{dy}{dx} = y$.

QUESTION 7. Argue that $\tan(\tan^{-1}(x)) = x$ for all real x, but that $\tan^{-1}(\tan(x))$ only holds for $-\pi/2 < x < \pi/2$.

QUESTION 8. Using an apropriate right-angled triangle, show that $\cos(\tan^{-} 1(2)) = 1/\sqrt{(}5)$.

QUESTION 9. Verify the following by calculating a suitable derivative.

(a) $\displaystyle\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\left(\frac{x}{a}\right)$ for $a > 0$

(b) $\displaystyle\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$