MA211 Calculus I – Problem Sheet 6

October 17, 2016, Lecturer: Claas Röver

QUESTION 1. Evaluate each of the following improper integrals if it exists, or justify that it diverges. *Hint:* Calculate $\frac{d}{dx}[\ln(\ln(x))]$.

(a) $\displaystyle\int_2^\infty \frac{1}{(x-1)^3}\,dx$ (b) $\displaystyle\int_0^\infty \frac{x}{1+2x^2}\,dx$ (c) $\displaystyle\int_3^\infty \frac{1}{x\ln(x)}\,dx$

QUESTION 2. (a) Carefully, state l'Hôpital's (or l'hospital's) rule.

(b) Use l'Hôpital's rule, possibly more than once, to find the following limits.

(i) $\displaystyle\lim_{x\to 0^+} \frac{\sin(x)}{x}$ (ii) $\displaystyle\lim_{x\to\infty} \frac{x^2}{e^x}$ (iii) $\displaystyle\lim_{x\to 0^+} x\ln(x)$

(c) Use the results of part (a) to evaluate the following improper integrals.

(i) $\displaystyle\int_0^{\pi/2} \frac{x\cos(x)-\sin(x)}{x^2}\,dx$ (ii) $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$ (iii) $\displaystyle\int_0^e \ln(x)\,dx$

QUESTION 3. Recall that a function is monotone increasing, if its derivative is positive.

(a) Show that $\ln(x) \le x-1$ for all $x \ge 1$.

(b) Using part (a), or otherwise, show that $\displaystyle\int_2^\infty \frac{1}{\ln(x)}\,dx$ and $\displaystyle\int_1^2 \frac{1}{\ln(x)}\,dx$ both diverge.

(c) Extending the above argument, show that $\displaystyle\int_4^\infty \frac{1}{\ln(\ln(x))}\,dx$ and $\displaystyle\int_e^4 \frac{1}{\ln(\ln(x))}\,dx$ both diverge.

QUESTION 4. For each of the following sequences, give the general formula for a_n, $n \ge 1$.

(a) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \dots$ (b) $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \dots$ (c) $-1, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \dots$

(d) $1, \frac{-1}{4}, \frac{1}{9}, \frac{-1}{16}, \frac{1}{25}, \dots$ (e) $1, 2, 6, 24, 120, 720, \dots$ (f) $1, 3, 9, 27, 81, 243, \dots$

QUESTION 5. For each of the following series, decide whether it converges or diverges.

(a) $\displaystyle\sum_{n=1}^\infty \frac{n}{n+1}$ (b) $\displaystyle\sum_{n=1}^\infty \frac{2n}{n^2+n+1}$ (c) $\displaystyle\sum_{n=1}^\infty \frac{(-1)^n}{n^2}$