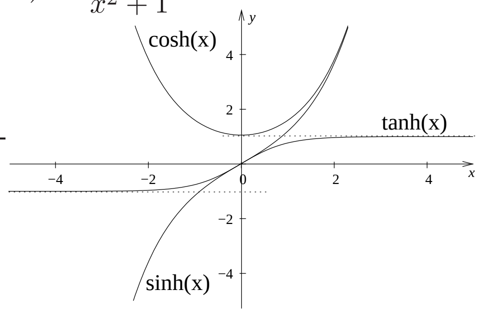
MA211 Calculus — Course Summary

Inverse function: Suppose $\mathcal{D} \subseteq \mathbb{R}$ and $f\colon \mathcal{D} \to \mathbb{R}$ is injective (on $\mathcal{D}$) with image $\mathcal{I}$. Then f is invertible and its inverse function $f^{-1}\colon \mathcal{I} \to \mathcal{D}$ satisfies $f^{-1}(f(x)) = x$ for all $x \in \mathcal{D}$, $f(f^{-1}(x)) = x$ for all $x \in \mathcal{I}$ and, if f is differentiable, then so is f^{-1} and $(f^{-1})'(x) = \dfrac{1}{f'(f^{-1}(x))}$ for all $x \in \mathcal{I}$.

Useful examples are $\dfrac{d}{dx}\left(\sin^{-1}(x)\right) = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d}{dx}\left(\tan^{-1}(x)\right) = \dfrac{1}{x^2+1}$.

Hyperbolic functions, their inverses and derivatives are given in the table below and sketched on the right.

$f(x)$	$f^{-1}(x)$	$f'(x)$
$\sinh(x) = \frac{1}{2}\left(e^x - e^{-x}\right)$	$\ln\left(x + \sqrt{x^2+1}\right),\ x \in \mathbb{R}$	$\cosh(x)$
$\cosh(x) = \frac{1}{2}\left(e^x + e^{-x}\right)$	$\ln\left(x + \sqrt{x^2-1}\right),\ x \geq 1$	$\sinh(x)$
$\tanh(x) = \frac{\sinh(x)}{\cosh(x)}$	$\frac{1}{2}\ln\left(\frac{1+x}{1-x}\right),\ -1 < x < 1$	$\frac{1}{\cosh^2(x)}$



A **sequence** $(a_n)_{n\geq 1} = (a_1, a_2, \ldots)$ *converges* to the limit $L \in \mathbb{R}$, if for every $\varepsilon > 0$, there exists $N \geq 1$ such that $|L - a_n| < \varepsilon$ for all $n \geq N$. A sequence *diverges*, if it does not converge.

Improper integrals have (at least) one boundary at $\pm\infty$ or a vertical asymptote of the integrand in the closed interval between the boundaries. They are calculated as limits of proper integrals as in

$$\int_b^\infty f(x)\,dx = \lim_{R\to\infty} \int_b^R f(x)\,dx \qquad \text{or} \qquad \int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx,$$

where f is continuous on the interval $I = (a, \infty)$ but has a vertical asymptote at a, and $b \in I$.

For $p \geq 0$ and $a > 0$, the <u>*p-integral formulæ*</u>, which you should be able to prove, are:

$$\int_a^\infty x^{-p}\,dx \ \text{ is } \begin{cases} \text{convergent}, & \text{if } p > 1 \\ \text{divergent}, & \text{if } p \leq 1 \end{cases}, \quad \text{and} \quad \int_0^a x^{-p}\,dx \ \text{ is } \begin{cases} \text{convergent}, & \text{if } p < 1 \\ \text{divergent}, & \text{if } p \geq 1 \end{cases}.$$

L'Hôpital's rule says $\displaystyle\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$, if $\displaystyle\lim_{x\to a} f(x)$ and $\displaystyle\lim_{x\to a} g(x)$ are both 0 or both $\pm\infty$, where $a \in \mathbb{R} \cup \{\pm\infty\}$.

Series are expressions of the form $\sigma = \displaystyle\sum_{n=0}^{\infty} a_n$, and defined as the limit of their sequence of *partial sums* $s_n = \displaystyle\sum_{j=0}^{n} a_j$, i.e. $\sigma = \displaystyle\lim_{n\to\infty} s_n$. *Geometric series* $\displaystyle\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ for $|r| < 1$. Convergence tests:

Null-sequence: If σ converges, then $\displaystyle\lim_{n\to\infty} a_n = 0$. Usually used as: if $\displaystyle\lim_{n\to\infty} a_n \neq 0$, then σ diverges.

Integral Test: If $a_n = f(n)$ with continuous f, then σ converges if and only if $\displaystyle\int_N^\infty f(x)\,dx$ does.

Put $\rho = \left\{ \begin{array}{ll} \displaystyle\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| & (\textit{Quotient Test}) \\ \displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} & (\textit{Root Test}) \end{array} \right\}$. Then $\left\{ \begin{array}{ll} \sigma \text{ diverges}, & \text{if } \rho > 1 \\ \sigma \text{ converges (absolutely)}, & \text{if } \rho < 1 \\ \text{no conclusion is possible}, & \text{if } \rho = 1 \end{array} \right\}$.

Differential equations are equations which involve a function $y = y(x)$, some of its derivatives and possibly x. An equation is of **order** n if the n-th derivative is the highest derivative that occurs.

<u>*First order separable*</u>: $\dfrac{dy}{dx} = f(x)g(y)$. Solve by integrating $\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$.

<u>*First order homogeneous*</u>: $\dfrac{dy}{dx} = f\left(\dfrac{y}{x}\right)$. With $v = \dfrac{y}{x}$ we get $\dfrac{dv}{dx} = \dfrac{f(v) - v}{x}$ which is separable.

<u>*First order linear*</u>: $\dfrac{dy}{dx} + p(x)y = q(x)$. With *integrating factor* $\mu = \displaystyle\int p\,dx$, we have $\dfrac{d\mu}{dx} = p$ and so $\dfrac{d}{dx}\left(ye^\mu\right) = e^\mu\left(\dfrac{dy}{dx} + py\right) = e^\mu q$. Hence the general solution is $y = e^{-\mu}\displaystyle\int qe^\mu\,dx$.

<u>*Second order linear, constant coefficients*</u>: $ay'' + by' + cy = 0$. The general solution is $Ae^{k_1x} + Be^{k_2x}$ or $Ae^{kx} + Bxe^{kx}$ or $e^{kx}(A\cos(\omega x) + B\sin(\omega x))$, depending on whether the *auxiliary equation* $ax^2 + bx + c = 0$ has two real solutions k_1, k_2 or one real solution k or two complex solutions $k + i\omega$.