

MA343 Group Theory I – Assignment 2

October 18, 2012, Lecturer: Claas Röver

QUESTION 1. Suppose $\alpha: G \rightarrow H$ is a surjective homomorphism of groups. Verify the following claims, where $g, \bar{g} \in G$ and $U \subset G$ is a subgroup.

- (a) The order of g^α divides the order of g . (Assume that every integer divides infinity.)
- (b) If g and \bar{g} commute, then so do g^α and \bar{g}^α .
- (c) The image of U , i.e. U^α , is a subgroup of H , and if U is normal in G , then U^α is normal in H .

QUESTION 2. Suppose U and V are subgroups of a group G .

- (a) Show that $U \cap V$ is also a subgroup of G .
- (b) Show that if V is normal in G , then $UV = \{uv \mid u \in U, v \in V\}$ is a subgroup of G .
- (c) Find an example to show that UV is not a subgroup in general.

QUESTION 3. (a) Let $D = G \times H$ be the direct product of groups G and H . Prove that D has a normal subgroup N , such that $N \cong G$ and $D/N \cong H$.

(b) Let U and V be subgroups of a group G . Then the following are equivalent.

- (i) $G \cong U \times V$
- (ii) $[U, V] = 1$, $U \cap V = 1$ and $UV = G$.

QUESTION 4. Suppose that G is a group with $g^2 = 1$ for all $g \in G$. Show that G is necessarily abelian. Prove that if G is finite, then $|G| = 2^k$ for some $k \geq 0$ and G needs at least k generators.

QUESTION 5. Consider S_5 , the symmetric group of degree five. Does it have a subgroup isomorphic to $C_5 \times C_5$? Does it have elements of order 6? Does it have a subgroup isomorphic to D_5 ? What about a subgroup isomorphic to D_6 ?

Hand in your solutions at the **beginning** of the lecture on **Friday, October 26, 2012**.