MA500 Advanced Group Theory – Problem Sheet 1

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Throughout this problem sheet, the field with p elements, for a prime p, is denoted $\mathbb{F}_p$. That is $\mathbb{F}_p \cong \mathbb{Z}/p\mathbb{Z}$ as a ring.

Furthermore, for any field $\mathbb{F}$, the *general linear group of dimension* n *over* $\mathbb{F}$, $\mathrm{GL}_n(\mathbb{F})$, is defined by

$$\mathrm{GL}_n(\mathbb{F}) = \{A \mid A \text{ is an } n \times n \text{ matrix with entries from } \mathbb{F} \text{ and } \det(A) \neq 0\}.$$

TASK 1. Prove that, for any field $\mathbb{F}$, the determinant map

$$\begin{array}{rccc} \det\colon & \mathrm{GL}_n(\mathbb{F}) & \longrightarrow & \mathbb{F}^* \\ & A & \longmapsto & \det(A) \end{array}$$

is a group homomorphism, where $\mathbb{F}^*$ denotes the multiplicative group of the non-zero elements of $\mathbb{F}$. Hence, or otherwise, show that $\mathrm{SL}_n(\mathbb{F})$ is a normal subgroup of $\mathrm{GL}_n(\mathbb{F})$.

TASK 2. By arguing about the possibilities of choosing a set of n linearly independent vectors in $\mathbb{F}_p^n$, prove that the order of $\mathrm{GL}_n(\mathbb{F}_p)$ is $\displaystyle\prod_{k=0}^{n-1}(p^n - p^k)$. Then deduce that $|\mathrm{SL}_n(\mathbb{F}_p)| = |\mathrm{GL}_n(\mathbb{F}_p)|/(p-1)$.

TASK 3. Show that S_4, i.e. the symmetric group on four points, and $\mathrm{SL}_2(\mathbb{F}_3)$ have the same order but are not isomorphic. *Hint:* Look at their centres.
Find a composition series for $\mathrm{SL}_2(\mathbb{F}_3)$. Is it a solvable group?

TASK 4. Describe the non-trivial semi-direct product of two infinite cyclic groups $G = C_\infty \ltimes C_\infty$. Is G solvable? Justify your answer. What is the isomorphism type of the centre of G?

TASK 5. Consider the matrices

$$I = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \qquad \text{and} \qquad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

over the complex numbers $\mathbb{C}$, where $i^2 = -1$, as usual. Determine the isomorphism type of the subgroup Q of $\mathrm{SL}_2(\mathbb{C})$ generated by I and K. Find a composition series for Q and also the derived series of Q. Finally, determine $\mathrm{Aut}(Q)$.