MA500 Advanced Group Theory – Problem Sheet 2

January 25, 2016, Lecturer: Claas Röver

TASK 1. Let $\mathbb{F}$ be a field.

(a) Show that the matrices of the form $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix}$ with $x, y \in \mathbb{F}$ form a subgroup of $\mathrm{SL}_3(\mathbb{F})$ isomorphic to $\mathbb{F}^2$, the aditive group of the 2-dimensional vector space over $\mathbb{F}$.

(b) Verify that

$$\begin{array}{rccc} \phi: & \mathrm{GL}_2(\mathbb{F}) & \longrightarrow & \mathrm{GL}_3(\mathbb{F}) \\ & A & \longmapsto & \begin{pmatrix} \begin{matrix} & A & \end{matrix} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix} \end{array}$$

is an injective group homomorphism.

(c) Use parts (a) and (b) to prove that the semi-direct product of $\mathbb{F}^2$ with $\mathrm{GL}_2(\mathbb{F})$ under its natural action on 2-dimensional vectors over $\mathbb{F}$ is isomorphic to a subgroup of $\mathrm{GL}_3(\mathbb{F})$.

(d) Does the same construction work to embed $\mathrm{GL}_n(\mathbb{F})$ acting on n-dimensional vectors into $\mathrm{GL}_{n+1}(\mathbb{F})$? Briefly justify your answer.

TASK 2. Fix $n \geq 2$ and let $U = U_n(\mathbb{F})$ be the group of upper triangular $n \times n$ matrices over the field $\mathbb{F}$ with ones on the diagonal. Find the derived series of U and hence decide whether U is solvable.

TASK 3. In the lectures, an abstract argument was given that $\mathcal{D}_4$, the dihedral group of order eight, is isomorphic to $\mathcal{C}_2 \wr \mathcal{C}_2$. Write down such an isomorphism explicitely. It is enough to specify the images of the elements of a generating set and prove that this partial map extends (uniquely) to an injective homomorphism. Why do you not have to prove surjectivity?

TASK 4. Let G and H be groups. Show that the centre of the unrestricted standard wreath product of G with H, is isomorphic to the centre of G, i.e. $Z(G \wr H) \cong Z(G)$, but that the centre of the restricted standard wreath product of G with H is trivial if H is infinite, i.e. $Z(G \bar{\wr} H) = 1$ when $|H| = \infty$. Note: If $|H| < \infty$, then $G \wr H = G \bar{\wr} H$.

TASK 5. Let $\mathbb{C}$ denote the complex numbers and let p be a prime. Define

$$\mathcal{C}_{p^\infty} = \{z \in \mathbb{C} \mid z^{p^k} = 1 \text{ for some } k \in \mathbb{N}\}.$$

In other words, $\mathcal{C}_{p^\infty}$ is the set of all p^k-th roots of unity in $\mathbb{C}$. Prove that $\mathcal{C}_{p^\infty}$ with the induced multiplication from $\mathbb{C}$ is a group that is not finitely generated. What are the finitely generated subgroups of $\mathcal{C}_{p^\infty}$? What kind of quotients does $\mathcal{C}_{p^\infty}$ have?