MA500 Advanced Group Theory – Problem Sheet 3

March 1, 2017, Lecturer: Claas Röver

TASK 1. **Ping-Pong Lemma** Assume that the group G acts faithfully on a set X, i.e. for every $g \in G$ there exists an $x \in X$ with $x^g \neq x$. Assume further that A and B are non-trivial subgroups of G, of which one has at least three elements, and that Y and Z are disjoint subsets of X such that for all non-trivial $a \in A$ and all non-trivial $b \in B$ we have $Z^a \subset Y$ and $Y^b \subset Z$. Prove that the subgroup generated by A and B is the free product $A * B$.
Hint: It suffices to show that every (free product) reduced word acts non-trivially. Note also that in order to show that a word w is non-trivial, you can show that a conjugate of w is non-trivial.

TASK 2. Let $G_1 = \mathcal{A}_4$ be an alternating group on four points, let $G_2 = \langle s, t \mid s^2, t^2, (st)^4 \rangle \cong \mathcal{D}_4$ be a dihedral group of order eight, and let $H = \langle x, y \mid x^2, y^2, [x, y] \rangle \cong \mathcal{C}_2 \times \mathcal{C}_2$ be a Klein-four group.

(a) Specify injective homomorphisms

$$\varphi_1 \colon H \longrightarrow G_1 \quad \text{and} \quad \varphi_2 \colon H \longrightarrow G_2$$

by defining the images of the generators of H and verifying that then the relators of H are mapped to the identity.

(b) Using the standard representation of G_1 by permutations on $\{1, 2, 3, 4\}$, find a transversal T_1 for H^{φ_1} in G_1.

(c) Find a transversal T_2 for H^{φ_1} in G_2.

Now let $G = G_1 *_H G_2$ be the free product of G_1 and G_2 with H amalgamated according to the embeddings φ_1 and φ_2 from part (a).

(d) Write down a presentation for G. *Hint:* You can use the fact that G_1 is a semi-direct product of H^{φ_1} with a cyclic group of order three to find a presentation of G_1.

(e) Carefully show how to rewrite the element

$$(123)sts(24)(13)st(134)tst(12)(34)$$

into normal form with respect to the transversals T_1 and T_2 form parts (b) and (c).

(f) Give, and briefly justify, examples of non-trivial elements g and g' of G which are not contained in $G_1 \cup G_2$ such that g has infinite order, while g' has finite order.

TASK 3. Let G and H be non-trivial finite groups with G of order at least three. Show that the free product $G * H$ has a subgroup isomorphic to the free group of rank two.
Hint: Give the two generators and show that every freely reduced word over those generators is also reduced in the free product.