MA500 Advanced Group Theory – Assignment 2

March 20, 2017, Lecturer: Claas Röver

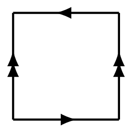
Hand in your solution at the beginning of the lecture on **Monday, 27 March 2017**.

TASK 1. Recall that for subsets S and T of a group, we defined $ST = \{st \mid s \in S,\ t \in T\}$.

(a) Let G be any group. Show that for a normal subgroup N and any subgroup K of G we have $NK = KN$ and that KN is a subgroup of G.

(b) Give an example of a group G and two subgroups H and K of G such that HK is not a subgroup of G.

TASK 2. The Klein bottle is a non-orientable closed surface obtained from a square by identifying the edges as in the figure below.



(a) Write down a presentation for the fundamental group of the Klein bottle that comes from this identifiaction.

(b) Many books claim that the fundamental group of the Klein bottle has the presentation $\langle x, y \mid x^2y^2 = 1\rangle$. Prove that your presentation and this one give isomorphic groups. *Hint:* Consider a diagonal of the square.

(c) It is a fact that every non-orientable closed surface Σ' is the quotient of an orientable closed surface Σ such that every point of Σ' has precisely two preimages in Σ, i.e. Σ is a two-sheeted cover of Σ.

Find the orientable closed surface that is a two-sheeted cover of the Klein bottle. *Hint:* Think about the fundamental domain the surface should have so that it can cover the square above.

TASK 3. Recall from the lectures, that the hyperbolic plane $\mathbb{H}^2$ can be viewed as points on the hyperboloid in $\mathbb{R}^3$, that is

$$\mathbb{H}^2 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_2^2 - x_3^2 = 1,\ x_1 > 0\}.$$

Show that $d(x, y) = \cosh^{-1}(B(x, y))$ defines a metric on $\mathbb{H}^2$, where $B(x, y) = x_1y_1 - x_2y_2 - x_3y_3$ and $\cosh^{-1}(r) = \ln\left(r + \sqrt{r^2 - 1}\right)$ $(r \geqslant 1)$ is the inverse function of the hyperbolic cosine $\cosh(r) = \frac{1}{2}(e^r + e^{-r})$ $(r \geqslant 0)$.

TASK 4. Write down the solution to Task 2 on Problem Sheet 3.

TASK 5. Prove that an element w of the free product $G * H$ has finite order if and only if it is conjugate to an element of finte order in G or H.