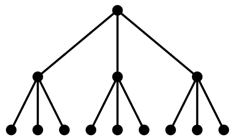
MA500 Advanced Group Theory – Revision Problems

March 28, 2017, Lecturer: Claas Röver

TASK 1. Let $G = S_3 \wr_X S_3$, the permutational wreath product of two symmetric groups of degree three, i.e. the automorphism group of the tree depicted below.



(a) Give a presentation for G and determine the isomorphism type of $G/[G, G]$.

(b) Find a composition series for G. Which composition factors do occur and how often?

(c) Using the action of G to describe an embedding of G into S_9, the symmetric group of degree nine; you can use the generators from part (a).

(d) Can G be a subgroup of S_k for $k \leqslant 8$? Justify your answer.

TASK 2. Let G be a group and H be a subgroup of G. Recall the definitions of the *centraliser of* H *in* G and the *normaliser of* H *in* G, denoted $C_G(H)$ and $N_G(H)$, respectively:

$$C_G(H) = \{g \in G \mid [g, h] = 1 \text{ for all } h \in H\} \quad \text{and} \quad N_G(H) = \{g \in G \mid H^g = H\}.$$

(a) Show that H has precisely $|G : N_H(G)|$ distinct conjugates in G.

(b) Show that $C_G(H)$ is a normal subgroup of $N_G(H)$.

(c) In $\mathcal{D}_8$, the dihedral group of order 16, find all possible subgroups together with their centralisers and normalisers.

(d) Show that $\mathrm{FC}(G) = \{g \in G \mid |G : C_G(\langle g \rangle)| < \infty\}$, that is the set of elements of G whose centraliser has finite index in G, is a normal subgroup of G.

TASK 3. Suppose G is a group and that $\alpha \colon G \longrightarrow A$ and $\beta \colon G \longrightarrow B$ are surjective homomorphisms with $\ker(\alpha) \cap \ker(\beta) = 1$.

(a) Prove that G is isomorphic to a subgroup of $A \times B$.

(b) Find an example which shows that G is not necessarily isomorphic to $A \times B$.

(c) Show that $G \cong A \times B$ if and only if $\ker(\alpha)\ker(\beta) = G$.

TASK 4. Let $G = \mathrm{Sym}(X)$, where X is a countably infinite set. The right regular action of a group on itself shows that G contains (an isomorphic copy of) every countable group.

(a) Show that G is not finitely generated.

(b) The *support of* $g \in G$ is defined as $\mathrm{supp}(g) = \{x \in X \mid x^g \neq x\}$. Prove that $F = \{g \in G \mid |\mathrm{supp}(g)| < \infty\}$ is a countable subgroup of G.

(c) Show that every finitely generated subgroup of F is finite, and hence that F is not finitely generated.

(d) Prove that F is simple. *Hint:* Remember that $\mathrm{Alt}(Y)$ is simple for $5 \leqslant |Y| < \infty$.

TASK 5. Let $G = C_2 \wr_R C_\infty$ be the standard restricted wreath product of a cyclic group of order two with an infinite cyclic group. Verify that $\langle a, t \mid [a, a^{t^i}] = a^2 = 1 \rangle$ is a presentation for G. Describe a normal form and a procedure transforming every word into normal form with respect to the generators a and t.