

Shift Actions on 2–cocycles

by

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3.4 Finite abelian groups

To help us better understand the quotient modules $\{K[G]^{(j)}\}$, we analyse them in the case when G is an abelian group; in particular, when G is a cyclic group and when G is an elementary abelian p -group. There are three reasons why these two special cases are analysed.

First, the case where G is cyclic is quite elegant and self-contained, as we shall see in Section 4.3. Even when $K[\mathbb{Z}_v]$ is neither nilpotent nor semisimple, we can construct a complete enumerative analysis of the \mathbb{Z}_v -orbit structure of all quotient modules $K[\mathbb{Z}_v]^{(j)}$.

Second, in our enumerative analyses of Chapter 4, we will see evidence that the group algebras $K[\mathbb{Z}_{p^r}]$ and $K[\mathbb{Z}_p^r]$ may be opposite extremes in our study of all group algebras where the group is of order p^r . This idea was hinted at in Subsection 3.3.3, and will be left as an open question in Subsection 4.5.1

Finally, in Chapter 5, we will see the link between the G -action on $K[G]^{(2)}$ and the shift action on $B^2(G, K)$. In the existing literature, the analysis or application of shift actions on $Z^2(G, K)$ assumes G is abelian. In fact, the shift action where G is cyclic is analysed in [30] and some cases when G is a small elementary abelian p -group appear in [28].

We first look at what happens when G is not necessarily a p -group.

3.4.1 On groups with complemented Sylow subgroups

Suppose G is a group with a Sylow p -subgroup which has a complement in G . In other words, $|G| = p^r b$, where p does not divide b , and $G = PQ$, where P is a subgroup of order p^r , where Q is a subgroup of order b and where $[P, Q] = \{1\}$ (in other words, $G \cong P \times Q$). For example, G can be any finite abelian group. By Lemma 1.3, we have a natural isomorphism

$K[G]^{(0)} \cong K[P]^{(0)} \otimes_K K[Q]^{(0)}$ of $K[G]$ -modules. Moreover, by the basic properties of tensor products and by applying Lemma 3.4 to $K[Q]$, we have

$$K[G]^{(0)} \cong \underbrace{\left(K[P]^{(0)} \otimes_K K\widehat{Q} \right)}_{A_1:=} \oplus_K \underbrace{\left(K[P]^{(0)} \otimes_K \omega(K[Q]) \right)}_{A_2:=}$$

The submodule A_2 contains no G -stable elements since $\omega(K[Q])$ contains no Q -stable elements. Thus, the G -stable elements of $K[G]^{(0)}$ are in the submodule $A_1 = K[P]^{(0)} \otimes_K K\widehat{Q}$, and they are precisely of the form $\alpha \otimes \widehat{Q}$, where $\alpha \in {}^P K[P]^{(0)}$. It is straightforward to prove

$$K[G]^{(j)} \cong \left(K[P]^{(j)} \otimes_K K\widehat{Q} \right) \oplus_K \left(K[P]^{(0)} \otimes_K \omega(K[Q]) \right). \quad (3.13)$$

By Corollary 3.10, there exists $l \in \mathbb{N}$ such that $K[G]^{(k)} \cong K[P]^{(0)} \otimes_K \omega(K[Q])$ whenever $k > l$: take $l = l(P)$, the length of the power filtration of $K[P]^{(0)}$.

3.4.2 Cyclic Groups

Assume K has characteristic $p > 0$. We look at $K[\mathbb{Z}_w]^{(j)}$, where \mathbb{Z}_w is the cyclic group of order w . Suppose $w = p^r b$, where p does not divide b and where $r \geq 1$ (because if $r = 0$, then $K[\mathbb{Z}_w] = K[\mathbb{Z}_b]$ is semisimple and we have nothing new to say). Using the isomorphism $\mathbb{Z}_w \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_b$, equation (3.13) implies it suffices to look at the cyclic p -groups:

$$K[\mathbb{Z}_w]^{(j)} \cong \left(K[\mathbb{Z}_{p^r}]^{(j)} \otimes_K K\widehat{\mathbb{Z}_b} \right) \oplus_K \left(K[\mathbb{Z}_{p^r}]^{(0)} \otimes_K \left(K[\mathbb{Z}_b]/K\widehat{\mathbb{Z}_b} \right) \right).$$

We already know the dimension subgroups of \mathbb{Z}_{p^r} (generated by x): see Example 3.15. In fact, from Subsection 3.3.1, we know the power filtration of $\omega(K[G])$ has length $p^r - 1$, with

$$\dim_K \left(\omega(K[G])^t / \omega(K[G])^{t+1} \right) = 1 \text{ when } 0 \leq t < p^r.$$

The η -basis can be constructed as in Lemma 1.17 using $x_i = x^{p^{i-1}}$ for $i = 1, \dots, r - 1$, hence $\nu(x^{p^i}) = p^i$. For $t = 0$ to $p^r - 1$, there exists precisely

4.3.3 Cyclic groups in general

Let K be a finite field of characteristic p and order q . Consider $K[\mathbb{Z}_{p^r t}]$ where $r, t \in \mathbb{N}$ and $t > 1$ is not divisible by p .

Since $\mathbb{Z}_{p^r t} \cong \mathbb{Z}_{p^r} \times \mathbb{Z}_t$, we can use isomorphism (3.13):

$$K[\mathbb{Z}_{p^r t}]^{(j)} \cong K[\mathbb{Z}_{p^r}]^{(j)} \oplus_K (K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)}), \quad (4.2)$$

where $K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)}$ contains no nontrivial elements of orbit size $1, p, \dots, p^r$. Thus, columns $1, p, \dots, p^r$ of the orbit size table for $K[\mathbb{Z}_{p^r t}]$ are precisely columns $1, p, \dots, p^r$ of the orbit size table for $K[\mathbb{Z}_{p^r}]$; the unique injection of $K[\mathbb{Z}_{p^r}]^{(j)}$ into the right hand side of (4.2) is the inclusion map.

Now suppose s divides $p^r t$ and s is not a power of p . We want to count the elements with orbit size s in $K[\mathbb{Z}_{p^r t}]^{(j)}$, for $0 \leq j \leq p^r$. If s is not divisible by p^r , then we obtain the (j, s) entry from the orbit size table for $K[\mathbb{Z}_{p^{r-1} t}]$. Indeed, for $0 \leq i \leq p^r - p^{r-1}$, the Ω_i defined by (3.15) generalise as follows: if $k = \min\{j, p^r - p^{r-1}\}$ and $l = \max\{0, j - (p^r - p^{r-1})\}$, then define

$$\begin{aligned} \Omega_k \oplus (\Omega_0 \otimes \iota) : K[\mathbb{Z}_{p^{r-1}}]^{(l)} \oplus_K (K[\mathbb{Z}_{p^{r-1}}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)}) \\ \longrightarrow K[\mathbb{Z}_{p^r}]^{(j)} \oplus_K (K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)}) \end{aligned}$$

by

$$\alpha + (\beta \otimes \gamma) \longmapsto \Omega_k(\alpha) + (\Omega_0(\beta) \otimes \gamma),$$

where $\alpha \in K[\mathbb{Z}_{p^{r-1}}]^{(l)}$, $\beta \in K[\mathbb{Z}_{p^{r-1}}]^{(0)}$ and $\gamma \in K[\mathbb{Z}_t]^{(1)}$, and where ι is the identity map on $K[\mathbb{Z}_t]^{(1)}$. The map $\Omega_k \oplus (\Omega_0 \otimes \iota)$ is an injection onto all the $\langle x^{p^{r-1}} \rangle$ -stable elements in $K[\mathbb{Z}_{p^r}]^{(j)} \oplus_K (K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)})$. Indeed, if $s = p^a b$ where $0 \leq a < r$ and b divides t , it is straightforward to show that the (j, s) entry is $\chi_K^{(m)}(\mathbb{Z}_s)$, where $m = 0$ if $j \leq p^r - p^{r-1}$. It is straightforward to show:

$$\begin{aligned} m &= \max \{0, j - (p^r - p^{r-1}) - (p^{r-1} - p^{r-2}) - \cdots - (p^{a+1} - p^a)\} \\ &= \max \{0, j - p^r + p^a\}. \end{aligned}$$

Now suppose $s = p^r b$, where b divides t and, of course, $b \neq 1$ or t . Then we find entry (j, s) from the orbit size table for $K[\mathbb{Z}_{p^r b}]$. If, on the right hand side of (4.2), z generates \mathbb{Z}_t and y generates \mathbb{Z}_b , then

$$\begin{aligned} g : K[\mathbb{Z}_{p^r}]^{(j)} \oplus_K (K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_b]^{(1)}) \\ \longrightarrow K[\mathbb{Z}_{p^r}]^{(j)} \oplus_K (K[\mathbb{Z}_{p^r}]^{(0)} \otimes K[\mathbb{Z}_t]^{(1)}) \end{aligned}$$

defined by

$$g \left(\alpha + (\beta \otimes \pi_1(\sum_{i=0}^{b-1} a_i y^i)) \right) = \alpha + (\beta \otimes \pi_1(\widehat{\langle z^{t/b} \rangle} \cdot \sum_{i=0}^{b-1} a_i z^i))$$

is an injection onto the $\langle z^{t/b} \rangle$ -stable elements. Thus, entry (j, s) of the orbit size table for $K[\mathbb{Z}_{p^r b}]$ is $\chi_K^{(j)}(\mathbb{Z}_s)$.

We summarise the above results in the following theorem:

Theorem 4.8. *Suppose $0 \leq a \leq r$ and b divides t . Entry $(j, p^a b)$ of the orbit size table for $K[p^r t]$ is*

- (i) entry (j, p^a) of the orbit size table for $K[p^r]$ if $b = 1$;
 - (ii) $\chi_K^{(m)}(\mathbb{Z}_{p^a b})$, where $m = \max \{0, j - p^r + p^a\}$, if $0 \leq a < r$ and $1 < b \leq t$;
 - (iii) entry $(j, p^r b)$ of the orbit size table for $K[p^a b]$ if $a = r$ and $1 < b < t$,
- and, of course, entry $(j, p^r t)$ is $q^{p^r t - j}$ minus the sum of the rest of the entries.

□

To illustrate the theorem, we give the orbit size tables for $\mathbb{F}_2[\mathbb{Z}_2]$, $\mathbb{F}_2[\mathbb{Z}_3]$ and $\mathbb{F}_2[\mathbb{Z}_4]$, respectively:

$$\begin{pmatrix} 2 & 2 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 2 & 0 & 6 \\ 1 & 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 2 & 2 & 0 & 12 \\ 2 & 2 & 0 & 4 \\ 2 & 2 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{pmatrix}$$