Isolated Hadamard Matrices from Mutually Unbiased Product Bases

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Abstract

A new construction of complex Hadamard matrices of composite order $d = pq$, with primes p, q, is presented which is based on pairs of mutually unbiased bases containing only product states. For product dimensions $d < 100$, we illustrate the method by deriving many previously unknown complex Hadamard matrices. We obtain at least 12 new *isolated* matrices of Butson type, with orders ranging from 9 to 91.

1 Introduction

A square matrix H of order d is a complex Hadamard matrix if it is unitary, $HH^\dagger = I$, and if its elements have equal modulus. This definition generalises the concept of a *real* Hadamard matrix with matrix elements limited to the values $\pm 1/\sqrt{d}$. The first known construction of such matrices is due to J.J. Sylvester [1] while they take their name from J. Hadamard who found that the absolute value of the determinant of a unitary matrix achieves its maximum if all its matrix elements have the same modulus [2].

Since then, complex Hadamard matrices have made their appearance in various branches of both mathematics and physics. For example, they relate to the problems of finding bi-unitary sequences and cyclic n-roots [3], they can be useful in constructing certain *-subalgebras of finite von Neumann algebras [4], and error correcting codes [5]. They also have applications in quantum information, representing an important ingredient in teleportation and dense coding schemes [6], and they are closely linked to mutually unbiased bases [7]. For a detailed overview of their applications, see [5, 8].

In view of their many uses, a complete classification of complex Hadamard matrices would be highly desirable but has not yet been achieved. All complex Hadamard matrices, up to equivalence, are known for dimensions $d \leq 5$ [9, 10], but their classification remains

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incomplete for higher dimensions. Many continuous families of complex Hadamard matrices already exist for $d = 6$, including the three- and four-parameter families found in [11] and [12], respectively; in addition, a single isolated complex Hadamard matrix, disconnected from any family has been found [13,14]. A matrix is isolated if its *defect* – an upper bound on the dimensionality of the set of Hadamard matrices stemming from the matrix – is zero [15].

General construction methods exist in composite dimensions [16,17], and continuous families of complex Hadamard matrices have been obtained from so-called *parameterisations* of known Hadamard matrices [17–20]. There has also been some success in finding continuous families of complex Hadamard matrices for certain prime dimensions [21]. A survey of known complex Hadamard matrices is given in [15] for $d \leq 16$, with an updated online catalogue provided by [22].

In this paper we introduce a new technique to construct complex Hadamard matrices of order $d = pq$, $p \leq q$, with p and q prime, based on a link between Hadamard matrices and *pairs* of MU bases consisting entirely of *product* states. A $(d \times d)$ complex Hadamard matrix H is unitary, which is another way of saying that its columns represent orthonormal vectors of a basis in a Hilbert space of dimension d. By choosing its matrix elements to have modulus $1/\sqrt{d}$, the columns of H will form a basis which is *mutually unbiased* (MU, for short) to the standard basis, since the squared modulus of the inner product of any vector from H with any vector from the standard basis equals $1/d$. In other words, the matrices $\{I, H\}$, where I is the $(d \times d)$ identity matrix, represent a pair of MU bases. Since any pair of MU bases can be brought into *standard* form, which contains the standard basis, the classification problem of complex Hadamard matrices of order d is equivalent to the problem of finding all *pairs* of MU bases in the Hilbert space $\mathbb{C}^d$.

The method we use to construct Hadamard matrices originates from earlier studies of *pairs* of MU *product* bases in dimension six [23–25]. Generalising this method from $d = 2 \times 3$ to composite dimensions $d = pq$, we establish a general construction method resulting in previously undiscovered complex Hadamard matrices.

The paper is organised as follows: In Section 2 we summarise those properties of complex Hadamard matrices which we will use in later sections. Section 3 includes our first main result, Theorem 3, which describes a general construction for complex Hadamard matrices of size $d = pq$. In Section 4 we apply Theorem 3 to find new complex Hadamard matrices for $d \leq 15$. We briefly touch upon higher dimensions as well as potential generalisations of the construction in Section 5. Section 6 contains a summary of our results.

2 Complex Hadamard matrices

2.1 MU bases and Hadamard matrices

Two orthonormal bases $\mathcal{B} = \{|\phi_i\rangle\}$ and $\mathcal{B}' = \{|\psi_j\rangle\}$ in a Hilbert space $\mathbb{C}^d$ are MU if and only if $|\langle\phi_i|\psi_j\rangle|^2 = 1/d$ for all $i, j = 1\ldots d$. In prime and prime power dimensions p^n, $n \in \mathbb{N}$, complete sets of $(p^n + 1)$ MU bases exist [26, 27]. In *composite* dimensions such as $d = 6, 10, 12, ...,$ the question of whether complete sets of $(d + 1)$ MU bases exist remains open [28].

MU bases in a finite-dimensional Hilbert space $\mathbb{C}^d$ are closely related to complex Hadamard matrices. Given a set of $(r + 1)$ MU bases in standard form $\{\mathcal{I}, \mathcal{B}_1, \ldots \mathcal{B}_r\}$, with $\mathcal{I}$ the standard basis, the bases $\mathcal{B}_1, \ldots, \mathcal{B}_r$ are represented by $(d \times d)$ complex Hadamard matrices, $H_1, \ldots, H_r$. Here the vectors of each basis are given by the columns of the respective matrix. Since these matrices are MU to the identity matrix, their matrix elements have modulus $1/\sqrt{d}$.

Two complex Hadamard matrices H and K are *equivalent* to each other if one can write $H = D_1 P_1 K P_2 D_2$, where D_1, D_2 are diagonal matrices with diagonal elements consisting of phase factors and P_1, P_2 are permutation matrices. As a consequence, every complex Hadamard matrix can be *dephased* such that the matrix elements of its first row and column equal $1/\sqrt{d}$.

A useful criterion to determine whether two complex Hadamard matrices are *inequivalent* is to calculate their *Haagerup set*, defined in [9].

Definition 1. *The Haagerup set of a complex Hadamard matrix H of order n with matrix elements h_{ij} is given by*

$$\Lambda(H) = \{h_{ij}h_{kl}\bar{h}_{il}\bar{h}_{kj} : i, j, k, l = 1, \ldots, n\}, \tag{1}$$

where $\bar{h}_{ij}$ denotes the complex conjugate of h_{ij}.

Two matrices with *different* Haagerup sets are *inequivalent* since the set $\Lambda(H)$ is invariant under equivalences. Note, however, that two inequivalent matrices may have the same Haagerup set.

Sets of MU bases are *equivalent* if they can be mapped to each other by any of the following unitary or anti-unitary equivalence transformations: a unitary map acting on all bases within a set; the multiplication of states within bases by arbitrary phase factors; the permutation of states within a basis, and the complex conjugation of all bases. In addition, one may exchange any two bases within a set.

It is often the case that a single Hadamard matrix is contained in a subset of a larger family of complex Hadamard matrices of which there are two types: *affine* and *non-affine*

families. The following definition, given in [15], uses the notation ∘ as the entrywise product of matrices (also known as their Hadamard product), and EXP(.) denotes the entrywise exponential function acting on a matrix.

Definition 2. *Given a $(d \times d)$ complex Hadamard matrix H, an affine family of Hadamard matrices stemming from it is given by*

$$H(\mathcal{R}) = \{H \circ \mathrm{EXP}(i \cdot R) : R \in \mathcal{R}\}, \tag{2}$$

where $\mathcal{R}$ is a subspace of all real $(d \times d)$ matrices with zeros in the first row and column.

A family of Hadamard matrices is called *non-affine* if it cannot be written in this form.

The *defect $d(H)$* of a Hadamard matrix H, defined in [15], provides an upper bound on the dimensionality of any set of Hadamard matrices stemming from H. If a dephased Hadamard matrix has a defect of zero then the matrix is called *isolated*, expressing the fact that all complex Hadamard matrices in a neighbourhood of H are equivalent. To calculate the defect of H, we multiply all elements of the Hadamard matrix with independent phase factors, i.e. $H_{ij} \to e^{a_{ij}} H_{ij}$ for $i, j = 2 \ldots d$, and solve the set of equations, to first order, which are imposed by the unitarity condition (see [29] for an explicit example). For all but the smallest dimensions d or special cases, it seems imperative to use a computer program in order to determine the defect; the software we have used is MATLAB [30]. The defect provides only a weak upper bound on the dimensionality of a Hadamard family; higher-order solutions of the unitarity conditions often lead to stronger bounds [31].

2.2 Known constructions in composite dimensions

There are many known constructions of complex Hadamard matrices (cf. [22]) some of which apply only to specific dimensions. We briefly review two constructions of (affine) complex Hadamard matrices based on the tensor product of smaller matrices following [32].

Theorem 1. (Hosoya-Suzuki [16]) *Let $M_1, M_2, \ldots, M_v$ be $k \times k$, $N_1, N_2, \ldots, N_k$ be $v \times v$ complex Hadamard matrices. Then the generalised tensor product matrix, denoted by $Q = (M_1, M_2, \ldots, M_v) \otimes (N_1, N_2, \ldots, N_k)$, whose (i, j)th block is given by the matrix $Q_{ij} = diag([M_1]_{ij}, [M_2]_{ij}, \ldots, [M_v]_{ij})N_j$, is a complex Hadamard matrix of order vk.*

By using a simpler version of this tensor product structure, these matrices can be *parameterised*, i.e. embedded in larger families of Hadamard matrices.

Corollary 1. (Diţă [17]) *Let $M = (m_{ij})$ be a $k \times k$ and $N_1, N_2, \ldots, N_k$ be $v \times v$ dephased complex Hadamard matrices with m and $n_1, n_2, \ldots, n_k$ free parameters, respectively. Then the block matrix*

$$Q = \begin{pmatrix} m_{11}N_1 & m_{12}N_2 & \ldots & m_{1k}N_k \\ \vdots & & & \\ m_{k1}N_1 & m_{k2}N_2 & \ldots & m_{kk}N_k \end{pmatrix}, \tag{3}$$

with blocks $Q_{ij} = m_{ij}N_j$, is a complex Hadamard matrix of order vk with $m + \sum_{i=1}^{k} n_i + (k-1)(v-1)$ free parameters.

Any matrix which can be derived from Corollary 1 is called a *Diţă-type* complex Hadamard matrix.

2.3 Butson-type complex Hadamard matrices

We finally recall a special class of Hadamard matrices called *Butson-Hadamard* matrices.

Definition 3. *A complex Hadamard matrix of order d is a Butson-Hadamard matrix $BH(d,r)$ if its elements are r^{th} roots of unity, apart from a factor $1/\sqrt{d}$.*

Note that some authors write $BH(r,d)$ instead of $BH(d,r)$. It is straightforward to show that a Hadamard matrix is (equivalent to one) of Butson-type $BH(d,r)$: once dephased, all its matrix elements must be r^{th} roots of unity.

The simplest examples of Butson-type matrices occur when $r = 2$; in this case the matrices $BH(d,2)$ are the set of $(d \times d)$ *real* Hadamard matrices. The existence of $BH(d,r)$ matrices for arbitrary values of d and r is still an open problem; it remains unknown, for example, if real Hadamard matrices of the form $B(4n,2)$ exist for all integers n. A summary of existing Butson-Hadamard matrices with fourth and sixth roots of unity can be found in [32] and the known $BH(d,r)$ matrices for $d \leq 16$ are given in [22]. There are also several existence theorems for $BH(d,r)$ matrices, e.g.

Theorem 2. (Butson [33]) *When p is prime, a $BH(2p,p)$ matrix can be constructed.*

For $p = 3$, the matrix $BH(6,3)$ turns out to be the isolated matrix S_6, which was also found independently in [13, 14]. We have derived the matrices $B_{10} \in BH(10,5)$ and $B_{14} \in BH(14,7)$ following Butson's method (see Appendix A) since they will be important in the present context and seem to be unavailable in the literature.

Butson-Hadamard matrices will appear in Sections 4 and 5, where we will derive previously unknown examples of complex Hadamard matrices of orders up to 91. Most of these examples cannot be constructed from Theorem 2.

3 Complex Hadamard matrices from pairs of MU product bases

The following theorem shows how to construct a complex Hadamard matrix of order $d = pq$ where p and q are both prime, using sets of MU bases in dimensions p and q.

Theorem 3. *Suppose that $K_0, \ldots, K_{p-1}$ and $L_0, \ldots, L_{p-1}$ are unitary matrices of order q such that $K_m^\dagger L_n$ are complex Hadamard matrices for all $m, n = 0, \ldots, p-1$, i.e. K_m is MU to L_n, and let $\alpha_{ij}/\sqrt{p}$ be the (i,j)th element of a complex Hadamard matrix M of order p, with $|\alpha_{ij}| = 1$. Then the block matrix H_{pq} given by*

$$H_{pq} = \frac{1}{\sqrt{p}} \begin{pmatrix} \alpha_{11} K_0^\dagger L_0 & \alpha_{12} K_0^\dagger L_1 & \ldots & \alpha_{1p} K_0^\dagger L_{p-1} \\ \alpha_{21} K_1^\dagger L_0 & \alpha_{22} K_1^\dagger L_1 & \ldots & \alpha_{2p} K_1^\dagger L_{p-1} \\ \alpha_{31} K_2^\dagger L_0 & \alpha_{32} K_2^\dagger L_1 & \ldots & \alpha_{3p} K_2^\dagger L_{p-1} \\ \vdots & & & \\ \alpha_{p1} K_{p-1}^\dagger L_0 & \alpha_{p2} K_{p-1}^\dagger L_1 & \ldots & \alpha_{pp} K_{p-1}^\dagger L_{p-1} \end{pmatrix} \tag{4}$$

is a complex Hadamard matrix of order pq.

The theorem follows easily from factorising the matrix H_{pq} such that $H_{pq} = B_1^\dagger B_2$, where

$$B_1 = \begin{pmatrix} K_0 & 0 & \ldots & 0 \\ 0 & K_1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & K_{p-1} \end{pmatrix} \tag{5}$$

and

$$B_2 = \begin{pmatrix} \alpha_{11} L_0 & \alpha_{12} L_1 & \ldots & \alpha_{1p} L_{p-1} \\ \alpha_{21} L_0 & \alpha_{22} L_1 & \ldots & \alpha_{2p} L_{p-1} \\ \vdots & \vdots & & \vdots \\ \alpha_{p1} L_0 & \alpha_{p2} L_1 & \ldots & \alpha_{pp} L_{p-1} \end{pmatrix}. \tag{6}$$

The column vectors of the unitary matrices B_1 and B_2 form a pair of MU bases since the block matrices K_m are MU to L_n, i.e. $K_m^\dagger L_n$ are complex Hadamard matrices for all $m, n = 0, \ldots, p-1$. Thus, by mapping B_1 to the identity matrix using the unitary transformation $B_1^\dagger$, the matrix B_2 is simultaneously mapped to $B_1^\dagger B_2 = H_{pq}$, which is a complex Hadamard matrix. This completes the proof of Theorem 3.

In fact, the matrices B_1 and B_2 correspond to a pair of MU *product* bases where the columns of B_1 and B_2 form the vectors of each basis. We can write the pair of matrices B_1 and B_2 as the orthonormal bases

$$\mathcal{B}_1 = \left\{ |0_z\rangle \otimes \mathcal{K}_0, \ |1_z\rangle \otimes \mathcal{K}_1, \ \ldots, \ |(p-1)_z\rangle \otimes \mathcal{K}_{p-1} \right\} \tag{7}$$

and

$$\mathcal{B}_2 = \left\{ |0_a\rangle \otimes \mathcal{L}_0, \ |1_a\rangle \otimes \mathcal{L}_1, \ \ldots, \ |(p-1)_a\rangle \otimes \mathcal{L}_{p-1} \right\}, \tag{8}$$

respectively, where $|m_z\rangle \otimes \mathcal{K}_m$ denotes the tensor product of a state $|m_z\rangle$ from the standard basis of $\mathbb{C}^p$ with all states from a basis $\mathcal{K}_m$ of the space $\mathbb{C}^q$ corresponding to the matrix K_m. Similarly, $|n_a\rangle \otimes \mathcal{L}_n$ is defined such that $|n_a\rangle$ is a state in $\mathbb{C}^p$ corresponding to the nth

column vector of the $(p \times p)$ Hadamard matrix M, and $\mathcal{L}_n$ is a basis of $\mathbb{C}^q$ corresponding to the matrix L_n. Thus, by mapping $\mathcal{B}_1$ to the standard basis, the columns of the second basis $\mathcal{B}_2$ form the complex Hadamard matrix H_{pq}.

To simplify the matrix H_{pq} we perform equivalence transformations on the pair of MU bases $\{\mathcal{B}_1, \mathcal{B}_2\}$ such that $\mathcal{K}_0$ and $\mathcal{L}_0$ are mapped to the standard and Fourier basis of $\mathbb{C}^q$ respectively, and the orthonormal basis $\{|0_a\rangle, \ldots, |(p-1)_a\rangle\}$ is mapped to the Fourier basis of $\mathbb{C}^p$, i.e. $K_0 \equiv I_q$, $L_0 \equiv F_q$ and $M \equiv F_p$, with I_q the $(q \times q)$ identity matrix, and F_p, F_q, being the Fourier matrices of order p and q respectively. Since B_1 is MU to B_2, the set $\{I_q, K_1, K_2, \ldots, K_{p-1}\}$ is MU to $\{F_q, L_1, \ldots, L_{p-1}\}$, and as a consequence, $L_1, \ldots, L_{p-1}$ are complex Hadamard matrices. We will continue to use the simplification $M \equiv F_p$, $K_0 \equiv I_q$ and $L_0 \equiv F_q$ throughout.

In the trivial case of $K_1 = \ldots = K_{p-1} = I_q$, one can choose each matrix L_n to be a $(q-1)$-parameter family DF_q where $D = \mathrm{diag}(1, e^{a_1^n}, \ldots, e^{a_{q-1}^n})$ for each $n > 0$. In this case, H_{pq} is a $(p-1)(q-1)$-parameter family of complex Hadamard matrices.

In the following section we will show that for certain choices of the $(q \times q)$ matrices $K_1, \ldots, K_{p-1}, L_1, \ldots, L_{p-1}$, the matrix H_{pq} given in Theorem 3 supplies new examples of Hadamard matrices. Most of the matrices we find will be isolated, which is sufficient to confirm that Theorem 3 produces matrices not of Diţă-type: every Diţă-type matrix is embedded within a family depending on at least $(k-1)(v-1)$ free parameters, with $k, v > 1$.

4   Examples: $d \leq 15$

We will now use the construction given in Theorem 3 to find complex Hadamard matrices of composite dimensions $d = pq$, with prime numbers $p \leq q$. In this section, we limit ourselves to matrices of order $d \leq 15$ with $p \leq q$. Larger dimensions and possible generalisations of the construction will be considered briefly in Sec. 5.

4.1   Dimension four

In dimension four, all inequivalent complex Hadamard matrices are given by the one-parameter family $F_4(a)$, $a \in [0, \pi]$ [9, 10]. We re-derive this family from Theorem 3 using the block matrix,

$$H_4 = \frac{1}{\sqrt{2}} \begin{pmatrix} F_2 & L_1 \\ K_1^\dagger F_2 & -K_1^\dagger L_1 \end{pmatrix}, \tag{9}$$

where

$$F_2 \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{10}$$

is the (2×2) Fourier matrix and L_1, K_1 are specific unitary matrices of order two: to apply Theorem 3 it is necessary that the set $\{I_2, K_1\}$ is MU to $\{F_2, L_1\}$.

If K_1 is chosen as the identity, then L_1 can take the form

$$L_1(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ e^{ia} & -e^{ia} \end{pmatrix}, \tag{11}$$

where the column vectors of L_1 are indeed MU to the standard basis; an overall phase factor has been removed using equivalence transformations. Thus, H_4 turns into a one-parameter family of complex Hadamard matrices,

$$H_4(a) = \frac{1}{\sqrt{2}} \begin{pmatrix} F_2 & L_1(a) \\ F_2 & -L_1(a) \end{pmatrix}. \tag{12}$$

By permuting rows it is easily shown that $H_4(a)$ is equivalent to the one-parameter Fourier family $F_4(a)$. One can exchange K_1 with L_1 but the resulting family is still equivalent to $F_4(a)$; no other choices are possible. Note that in the four-dimensional case, $F_4(a)$ is equivalent to the transposed Fourier family $(F_4(a))^T$, a relation that does not always hold for larger composite dimensions.

4.2   Dimension six

In a recent paper [25], pairs of MU product bases of the form $\mathcal{B}_1 = \{|0\rangle \otimes \mathcal{K}_0, |1\rangle \otimes \mathcal{K}_1\}$ and $\mathcal{B}_2 = \{|0_x\rangle \otimes \mathcal{L}_0, |1_x\rangle \otimes \mathcal{L}_1\}$ were shown to give rise to the transposed Fourier family of complex Hadamard matrices [15] and the isolated matrix $S_6 \in BH(6, 3)$ [13, 14, 33]. Here we rederive this result on the basis of Theorem 3 where we start with the following (6×6) matrix

$$H_6 = \frac{1}{\sqrt{2}} \begin{pmatrix} F_3 & L_1 \\ K_1^\dagger F_3 & -K_1^\dagger L_1 \end{pmatrix}. \tag{13}$$

The (3×3) Fourier matrix is given by

$$F_3 \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}, \tag{14}$$

with $\omega = e^{2\pi i/3}$ being a third root of unity, and K_1, L_1 are unitary matrices of order three. The bases $\mathcal{B}_1$ and $\mathcal{B}_2$ will be MU if the pair $\{I_3, K_1\}$ is MU to the pair $\{F_3, L_1\}$. A proof given in [24] limits the possible choices for the matrices K_1 and L_1 to just three: (i) $K_1 = I_3$; (ii) $L_1 = F_3$; (iii) all four matrices are pairwise MU.

If $K_1 = I_3$, the most general set of matrices satisfying the MU conditions is a two-parameter set,

$$L_1(a, b) = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ e^{ia} & \omega e^{ia} & \omega^2 e^{ia} \\ e^{ib} & \omega^2 e^{ib} & \omega e^{ib} \end{pmatrix}. \tag{15}$$

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Thus, H_6 becomes a two-parameter family of complex Hadamard matrices which is equivalent to the transposed Fourier family $(F_6^{(2)})^T$.

If all four matrices I_3, F_3, K_1 and L_1 are MU then K_1 and L_1 are given by

$$K_1 = H_y \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega & \omega^2 & 1 \\ \omega & 1 & \omega^2 \end{pmatrix} \quad \text{and} \quad L_1 = H_w \equiv \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ \omega^2 & 1 & \omega \\ \omega^2 & \omega & 1 \end{pmatrix}, \quad (16)$$

or vice versa. Here, $\{I_3, F_3, H_y, H_w\}$ is the complete set of MU bases in $\mathbb{C}^3$. The complex Hadamard matrix H_6 in (13) associated with $K_1 = H_y$ and $L_1 = H_w$ is equivalent to S_6, the only known isolated complex Hadamard matrix of order six.

Thus, we have indeed constructed the transposed Fourier family of complex Hadamard matrices and the isolated matrix S_6 from Theorem 3. There is a multitude of additional Hadamard matrices of order six, including three- and four-parameter families [11, 12], but none of these can be derived from Theorem 3.

4.3 Dimension nine

The online catalogue [22] lists three types of complex Hadamard matrices of order nine: the four-parameter Fourier family $F_9^{(4)}$, the isolated matrix N_9 [34], and the matrix B_9 which has a defect of two [34]. Three Butson-Hadamard matrices are among them, namely $F_3 \otimes F_3 \in BH(9,3)$, $F_9 \in BH(9,9)$ and $B_9 \in BH(9,10)$. Theorem 3 allows us to identify an additional isolated Butson-Hadamard matrix of the form $BH(9,6)$.

The matrix in Eq. (4), for $d = 9$, has the structure

$$H_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} F_3 & L_1 & L_2 \\ K_1^\dagger F_3 & \omega K_1^\dagger L_1 & \omega^2 K_1^\dagger L_2 \\ K_2^\dagger F_3 & \omega^2 K_2^\dagger L_1 & \omega K_2^\dagger L_2 \end{pmatrix}, \quad (17)$$

where $\omega = e^{2\pi i/3}$ is a third root of unity, and the matrices I_3 and F_3 are the (3×3) identity and Fourier matrix, respectively. The (3×3) matrices K_1, K_2, L_1 and L_2 must be chosen such that the set $\{I_3, K_1, K_2\}$ is MU to $\{F_3, L_1, L_2\}$.

In the six-dimensional case (cf. Sec 4.2), the choice of pairs $\{I, K_1\}$ and $\{F_3, L_1\}$ was limited to either two matrices having identical columns (up to column permutations) or all four matrices being MU. Similarly, the choices for the triples $\{I_3, K_1, K_2\}$ and $\{F_3, L_1, L_2\}$ is restricted to the following two possibilities: (i) three matrices within one triple are identical (up to column permutations); (ii) two matrices are identical in each triple and all four MU bases $\{I_3, F_3, H_y, H_w\}$ are used.

If all matrices in one triple are the same, i.e. $K_1 = K_2 = I_3$, then H_9 is equivalent to the transposed Fourier family $(F_9^{(4)})^T$ of order 9, which depends on four real parameters.

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Now suppose that $K_1 = I_3$ and $L_1 = F_3$. The only remaining choice for matrices K_2 and L_2, (if $K_2 \neq I_3$), that satisfy the MU conditions is $K_2 = H_y$ and $L_2 = H_w$ (or vice versa), with H_y and H_w defined in Eq. (16). Denoting the resulting matrix by S_9, we find

$$S_9 = \frac{1}{\sqrt{3}} \begin{pmatrix} F_3 & F_3 & H_w \\ F_3 & \omega F_3 & \omega^2 H_w \\ H_y^\dagger F_3 & \omega^2 H_y^\dagger F_3 & \omega H_y^\dagger H_w \end{pmatrix}. \tag{18}$$

After dephasing, the matrix S_9 takes the form

$$S_9 = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & 1 & \omega & \omega^2 & \omega^2 & 1 & \omega \\ 1 & \omega^2 & \omega & 1 & \omega^2 & \omega & \omega^2 & \omega & 1 \\ 1 & 1 & 1 & \omega & \omega & \omega & \omega^2 & \omega^2 & \omega^2 \\ 1 & \omega & \omega^2 & \omega & \omega^2 & 1 & \omega & \omega^2 & 1 \\ 1 & \omega^2 & \omega & \omega & 1 & \omega^2 & \omega & 1 & \omega^2 \\ 1 & \omega & \omega & \omega^2 & 1 & 1 & -\omega & -1 & -1 \\ 1 & \omega^2 & 1 & \omega^2 & \omega & \omega^2 & -\omega^2 & -1 & -\omega^2 \\ 1 & 1 & \omega^2 & \omega^2 & \omega^2 & \omega & -\omega^2 & -\omega^2 & -1 \end{pmatrix}, \tag{19}$$

where $\omega = e^{2\pi i/3}$ is a *third* root of unity. Due to the negative signs in the bottom right block, S_9 is a Butson-Hadamard matrix containing *sixth* roots of unity, i.e. $S_9 \in BH(9,6)$. We find the defect of this matrix to be $d(S_9) = 0$ implying that S_9 is *isolated*.

Proposition 1. *The matrix S_9 is inequivalent to $F_9^{(4)}$, B_9 and N_9.*

Since $F_9^{(6)}$ and B_9 have non-zero defects and N_9 contains only tenth roots of unity, it is clear that S_9 is inequivalent to any known complex Hadamard matrix in dimension $d = 9$. The only other isolated complex Hadamard matrix known for $d = 9$, i.e. N_9, was found by a numerical search in [34]. As far as we are aware, the matrix S_9 has not been published previously.

4.4 Dimension ten

The known Hadamard matrices in dimension ten include the Fourier family $F_{10}^{(4)}$ and its transpose $(F_{10}^{(4)})^T$, the family $D_{10}^{(8)}$ found by Diţă [20] and a family $D_{10}^{(3)}$ stemming from D_{10} [35]. There is also an isolated matrix N_{10A}, a family $N_{10B}^{(3)}$ found originally in [34] and parameterised in [36], and $G_{10}^{(1)}$ [36]. Furthermore, there are the Butson-Hadamard matrices $X_{10} \in BH(10,5)$ [37] and $W' \in BH(10,6)$ [38]. Within the continuous families, several Butson-type matrices exist: $D_{10} \in BH(10,4)$, $F_2 \otimes F_5 \simeq F_{10} \in BH(10,10)$, and those contained in $D_{10}^{(8)}$, e.g. $H_{10}^{(\omega)}$, $d_{10}^{(\omega)} \in BH(10,6)$ given in [20].

In the following, we construct a complex Hadamard matrix of Butson-type based on the block matrix

$$H_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_5 & L_1 \\ K_1^\dagger F_5 & -K_1^\dagger L_1 \end{pmatrix}, \tag{20}$$

where it is necessary that the pair of (5×5) matrices $\{I_5, K_1\}$ is MU to the pair of (5×5) matrices $\{F_5, L_1\}$. Here, I_5 is the identity matrix and F_5 the Fourier matrix,

$$F_5 \equiv \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}, \tag{21}$$

with $\omega = e^{2\pi i/5}$ a fifth root of unity. Assuming that K_1 and L_1 are not identical to I_5 and F_5, respectively, one choice is to require that the matrices within the set $\{I_5, F_5, K_1, L_1\}$ are pairwise MU.

In dimension five, the complete set of six MU bases can be written as

$$\{I_5, F_5, H_1, H_2, H_3, H_4\}, \tag{22}$$

where H_i are the complex Hadamard matrices of order five given by

$$H_1 = DF_5, \quad H_2 = D^2 F_5, \quad H_3 = D^3 F_5, \quad H_4 = D^4 F_5, \tag{23}$$

and with a diagonal matrix,

$$D = \mathrm{diag}(1, \omega, \omega^4, \omega^4, \omega). \tag{24}$$

This characterisation of the complete set of MU bases is based on a construction in [10]. By choosing $K_1 = H_3$ and $L_1 = H_4$, the matrix H_{10} becomes

$$S_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_5 & H_4 \\ H_3^\dagger F_5 & -H_3^\dagger H_4 \end{pmatrix}, \tag{25}$$

which has the dephased form

$$S_{10} = \frac{1}{\sqrt{10}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^4 & 1 & \omega & \omega^2 & \omega^3 \\
1 & \omega^2 & \omega^4 & \omega & \omega^3 & \omega & \omega^3 & 1 & \omega^2 & \omega^4 \\
1 & \omega^3 & \omega & \omega^4 & \omega^2 & \omega & \omega^4 & \omega^2 & 1 & \omega^3 \\
1 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^4 & \omega^3 & \omega^2 & \omega & 1 \\
1 & \omega^3 & \omega^2 & \omega^2 & \omega^3 & 1 & \omega & \omega^4 & \omega^4 & \omega \\
1 & \omega^2 & 1 & \omega^4 & \omega^4 & \omega^3 & \omega^2 & \omega^3 & \omega & \omega \\
1 & \omega & \omega^3 & \omega & 1 & \omega^2 & \omega^4 & \omega^3 & \omega^4 & \omega^2 \\
1 & 1 & \omega & \omega^3 & \omega & \omega^2 & \omega^2 & \omega^4 & \omega^3 & \omega^4 \\
1 & \omega^4 & \omega^4 & 1 & \omega^2 & \omega^3 & \omega & \omega & \omega^3 & \omega^2
\end{pmatrix}, \tag{26}$$

where $\omega = e^{2\pi i/5}$ is a *fifth* root of unity.

Proposition 2. *The matrix S_{10} is inequivalent to $F_{10}^{(4)}$, $(F_{10}^{(4)})^T$, $D_{10}^{(3)}$, $D_{10}^{(8)}$, N_{10A}, $N_{10B}^{(3)}$, $G_{10}^{(1)}$ and W'.*

The matrix S_{10} is found to be *isolated* and contains only fifth roots of unity, therefore it is inequivalent to any of the complex Hadamard matrices listed in the proposition. However, we have not been able to show whether it is equivalent (or not) to the isolated Butson-type matrix $X_{10} \in BH(10, 5)$ or the matrix B_{10} given in Appendix A.

A different choice of the matrices K_1 and L_1, for example, $K_1 = H_1$ and $L_1 = H_2$, leads to the block matrix

$$S'_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_5 & H_2 \\ H_1^\dagger F_5 & -H_1^\dagger H_2 \end{pmatrix}, \tag{27}$$

which in dephased form is given by

$$S'_{10} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^2 & \omega^3 & \omega^4 & 1 & \omega \\ 1 & \omega^2 & \omega^4 & \omega & \omega^3 & \omega^3 & 1 & \omega^2 & \omega^4 & \omega \\ 1 & \omega^3 & \omega & \omega^4 & \omega^2 & \omega^3 & \omega & \omega^4 & \omega^2 & 1 \\ 1 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega^4 & \omega^3 \\ 1 & \omega^4 & \omega & \omega & \omega^4 & -1 & -\omega & -\omega^4 & -\omega^4 & -\omega \\ 1 & \omega & 1 & \omega^2 & \omega^2 & -\omega^2 & -\omega & -\omega^2 & -1 & -1 \\ 1 & \omega^3 & \omega^4 & \omega^3 & 1 & -\omega^3 & -1 & -\omega^4 & -1 & -\omega^3 \\ 1 & 1 & \omega^3 & \omega^4 & \omega^3 & -\omega^3 & -\omega^3 & -1 & -\omega^4 & -1 \\ 1 & \omega^2 & \omega^2 & 1 & \omega & -\omega^2 & -1 & -1 & -\omega^2 & -\omega \end{pmatrix}, \tag{28}$$

with $\omega = e^{2\pi i/5}$ a fifth root of unity. This matrix is a member of the family $BH(10, 10)$ and, with a defect $d(S'_{10}) = 8$, the maximum dimension of any smooth manifold stemming from S'_{10} will be eight. Several other matrices of the form $BH(10, 10)$ exist but we are not able to determine whether S'_{10} is equivalent to any of them.

Any choice of K_1 and L_1 from the set of MU bases $\{F_5, H_1, H_2, H_3, H_4\}$ will result in a Butson-type matrix of the form $BH(10, 5)$ or $BH(10, 10)$. It would be interesting to see if the various combinations of K_1 and L_1 result in further new inequivalent complex Hadamard matrices.

4.5 Dimension fourteen

The known complex Hadamard matrices of order fourteen are the six-parameter Fourier family $F_{14}^{(6)}$ and its transpose $(F_{14}^{(6)})^T$, the family $D_{14}^{(5)}$ found in [35], and a set of isolated

matrices $L_{14X}^{(0)}$ for $X = A, B, C, \ldots, N$ found in [36]. In addition, there are several Diţă-type matrices, listed in [22], obtained from Diţă's method of Corollary 1 using the Fourier matrix F_2 and any Hadamard matrix of order seven.

The matrix we construct from Theorem 3 consists of four blocks,

$$H_{14} = \frac{1}{\sqrt{2}} \begin{pmatrix} F_7 & L_1 \\ K_1^\dagger F_7 & -K_1^\dagger L_1 \end{pmatrix}, \tag{29}$$

where the Fourier matrix of order seven is given by

$$F_7 = \frac{1}{\sqrt{7}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega \end{pmatrix}, \tag{30}$$

with $\omega = e^{2\pi i/7}$, and the (7×7) matrices K_1 and L_1 are chosen from the complete set of eight MU bases of the space $\mathbb{C}^7$.

Let us denote the complete set of MU bases by $\{I_7, F_7, H_1, H_2, H_3, H_4, H_5, H_6\}$ where

$$H_j = D^j F_7 \tag{31}$$

and

$$D = \mathrm{diag}(1, 1, \omega, \omega^3, \omega^6, \omega^3, \omega), \tag{32}$$

with $j = 1, \ldots, 6$. The diagonal D is based on the construction of a complete sets of MU bases in prime dimensions presented in [39]. By choosing $K_1 = H_1$ and $L_1 = H_2$, we find a complex Hadamard matrix which, after dephasing, reads explicitly

$$S_{14} = \frac{1}{\sqrt{14}} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\ 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & 1 \\ 1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & 1 & \omega^3 \\ 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 & \omega^5 & \omega^2 & \omega^6 & \omega^3 & 1 & \omega^4 & \omega \\ 1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & \omega^6 & \omega^4 & \omega^2 & 1 & \omega^5 & \omega^3 & \omega \\ 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^2 & \omega & 1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 \\ 1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & 1 & \omega^5 & \omega^5 & \omega^4 & \omega^2 & \omega^6 & \omega^2 & \omega^4 \\ 1 & 1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & \omega^4 & \omega^5 & \omega^5 & \omega^4 & \omega^2 & \omega^6 & \omega^2 \\ 1 & \omega^6 & \omega^6 & 1 & \omega^2 & \omega^5 & \omega^2 & \omega & \omega^3 & \omega^4 & \omega^4 & \omega^3 & \omega & \omega^5 \\ 1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & 1 & \omega^3 & \omega^3 & \omega^6 & \omega & \omega^2 & \omega^2 & \omega & \omega^6 \\ 1 & \omega^4 & \omega^2 & \omega & \omega & \omega^2 & \omega^4 & \omega^3 & 1 & \omega^3 & \omega^5 & \omega^6 & \omega^6 & \omega^5 \\ 1 & \omega^3 & 1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & \omega & \omega^6 & \omega^3 & \omega^6 & \omega & \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^5 & \omega^2 & 1 & \omega^6 & \omega^6 & \omega^4 & \omega^3 & \omega & \omega^5 & \omega & \omega^3 & \omega^4 \end{pmatrix}, \tag{33}$$

with $\omega = e^{2\pi i/7}$. This is an *isolated* Butson-type complex Hadamard matrix of the form $BH(14,7)$.

Proposition 3. *The matrix S_{14} is inequivalent to $F_{14}^{(6)}$, $(F_{14}^{(6)})^T$, $D_{14}^{(5)}$, $L_{14X}^{(0)}$ and every Diţă-type matrix given in [22].*

Since S_{14} and $L_{14X}^{(0)}$ contain different roots of unity they are inequivalent. All other known matrices listed in the proposition are contained in *families* of complex Hadamard matrices, thus, they are inequivalent to S_{14}. However, it is not known whether S_{14} is equivalent to the Butson-Hadamard matrix B_{14} given in Appendix A.

We can construct further complex Hadamard matrices by choosing different MU bases for K_1 and L_1, e.g. if $K_1 = H_1$ and $L_1 = H_4$, the resulting matrix in dephased form is

$$S_{14}' = \frac{1}{\sqrt{14}} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & 1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & 1 & \omega^2 \\
1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & \omega^5 & \omega & \omega^4 & 1 & \omega^3 & \omega^6 & \omega^2 \\
1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 & \omega^3 & 1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 \\
1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & 1 \\
1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^4 & \omega^3 & \omega^2 & \omega & 1 & \omega^6 & \omega^5 \\
1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & 1 & -\omega^3 & -\omega & -\omega & -\omega^3 & -1 & -\omega^6 & -1 \\
1 & 1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & -1 & -\omega^3 & -\omega & -\omega & -\omega^3 & -1 & -\omega^6 \\
1 & \omega^6 & \omega^6 & 1 & \omega^2 & \omega^5 & \omega^2 & -\omega^5 & -\omega^6 & -\omega^2 & -1 & -1 & -\omega^2 & -\omega^6 \\
1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & 1 & \omega^3 & -\omega^4 & -\omega^3 & -\omega^4 & -1 & -\omega^5 & -\omega^5 & -1 \\
1 & \omega^4 & \omega^2 & \omega & \omega & \omega^2 & \omega^4 & -\omega^4 & -\omega & -1 & -\omega & -\omega^4 & -\omega^2 & -\omega^2 \\
1 & \omega^3 & 1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & -\omega^5 & -1 & -\omega^4 & -\omega^3 & -\omega^4 & -1 & -\omega^5 \\
1 & \omega^2 & \omega^5 & \omega^2 & 1 & \omega^6 & \omega^6 & -1 & -1 & -\omega^2 & -\omega^6 & -\omega^5 & -\omega^6 & -\omega^2
\end{pmatrix}, \tag{34}$$

with $\omega = e^{2\pi i/7}$. This is a Butson-Hadamard matrix of the form $BH(14,14)$, and has a defect of 12. It is unknown if S_{14}' is equivalent to a $BH(14,14)$ matrix contained within an existing family of Hadamard matrices.

4.6 Dimension fifteen

The only known complex Hadamard matrices of order fifteen are the eight-parameter Fourier family $F_{15}^{(8)}$, stemming from the Fourier matrix $F_{15} \simeq F_3 \otimes F_5 \in BH(15,15)$, and the transposed Fourier family $(F_{15}^{(8)})^T$. To construct a new (15×15) complex Hadamard matrix by means of Theorem 3, we use the block matrix

$$H_{15} = \frac{1}{\sqrt{3}} \begin{pmatrix}
F_5 & L_1 & L_2 \\
K_1^\dagger F_5 & \alpha K_1^\dagger L_1 & \alpha^2 K_1^\dagger L_2 \\
K_2^\dagger F_5 & \alpha^2 K_2^\dagger L_1 & \alpha K_2^\dagger L_2
\end{pmatrix}, \tag{35}$$

where $\alpha = e^{2\pi i/3}$ is a third root of unity, I_5, F_5, K_1, K_2, L_1 and L_2 are (5×5) matrices including the identity matrix I_5 and the Fourier matrix F_5 defined in Eq. (21). The set $\{I_5, K_1, K_2\}$ is MU to the set $\{F_5, L_1, L_2\}$.

If we use the *complete* set of six MU bases of the space $\mathbb{C}^5$, $\{I_5, F_5, H_1, H_2, H_3, H_4\}$, corresponding to $K_1 = H_1$, $K_2 = H_2$, $L_1 = H_3$ and $L_2 = H_4$, as defined in Eq. (23), the resulting complex Hadamard matrix becomes

$$
S_{15} = \frac{1}{\sqrt{3}} \begin{pmatrix} F_5 & H_3 & H_4 \\ H_1^\dagger F_5 & \alpha H_1^\dagger H_3 & \alpha^2 H_1^\dagger H_4 \\ H_2^\dagger F_5 & \alpha^2 H_2^\dagger H_3 & \alpha H_2^\dagger H_4 \end{pmatrix}. \tag{36}
$$

Apart from a factor $1/\sqrt{15}$, its dephased form reads explicitly

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega^3 & \omega^6 & \omega^9 & \omega^{12} & \omega^9 & \omega^{12} & 1 & \omega^3 & \omega^6 & \omega^{12} & 1 & \omega^3 & \omega^6 & \omega^9 \\
1 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & \omega^6 & \omega^{12} & \omega^3 & \omega^9 & 1 & \omega^3 & \omega^9 & 1 & \omega^6 & \omega^{12} \\
1 & \omega^9 & \omega^3 & \omega^{12} & \omega^6 & \omega^6 & 1 & \omega^9 & \omega^3 & \omega^{12} & \omega^3 & \omega^{12} & \omega^6 & 1 & \omega^9 \\
1 & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & \omega^9 & \omega^6 & \omega^3 & 1 & \omega^{12} & \omega^{12} & \omega^9 & \omega^6 & \omega^3 & 1 \\
1 & \omega^{12} & \omega^3 & \omega^3 & \omega^{12} & -\omega^5 & -\omega^{14} & -\omega^{11} & -\omega^{14} & -\omega & -\omega^{10} & -\omega & -\omega^4 & -\omega^4 & -\omega \\
1 & \omega^3 & 1 & \omega^6 & \omega^6 & -\omega^2 & -\omega^8 & -\omega^2 & -\omega^{14} & -\omega^{14} & -\omega^4 & -\omega^{13} & -\omega^4 & -\omega^7 & -\omega^7 \\
1 & \omega^9 & \omega^{12} & \omega^9 & 1 & -\omega^8 & -\omega^{11} & -\omega^2 & -\omega^{11} & -\omega^8 & -\omega & -\omega^{13} & -\omega^7 & -\omega^{13} & -\omega \\
1 & 1 & \omega^9 & \omega^{12} & \omega^9 & -\omega^8 & -\omega^8 & -\omega^{11} & -\omega^{11} & -\omega^2 & -\omega & -\omega & -\omega^{13} & -\omega^7 & -\omega^{13} \\
1 & \omega^6 & \omega^6 & 1 & \omega^3 & -\omega^2 & -\omega^{14} & -\omega^{14} & -\omega^2 & -\omega^8 & -\omega^4 & -\omega^7 & -\omega^7 & -\omega^4 & -\omega^{13} \\
1 & \omega^6 & \omega^9 & \omega^9 & \omega^6 & -\omega^{10} & -\omega^{13} & -\omega^7 & -\omega^7 & -\omega^{13} & \omega^5 & \omega^{14} & \omega^{11} & \omega^{11} & \omega^{14} \\
1 & \omega^9 & 1 & \omega^3 & \omega^3 & -\omega^7 & -\omega^4 & -\omega^7 & -\omega & -\omega & \omega^8 & \omega^{14} & \omega^8 & \omega^5 & \omega^5 \\
1 & \omega^{12} & \omega^6 & \omega^{12} & 1 & -\omega^{13} & -\omega^4 & -\omega^4 & -\omega & -\omega^4 & \omega^2 & \omega^5 & \omega^{11} & \omega^5 & \omega^2 \\
1 & 1 & \omega^{12} & \omega^6 & \omega^{12} & -\omega^{13} & -\omega^{13} & -\omega^4 & -\omega & -\omega^4 & \omega^2 & \omega^2 & \omega^5 & \omega^{11} & \omega^5 \\
1 & \omega^3 & \omega^3 & 1 & \omega^9 & -\omega^7 & -\omega & -\omega & -\omega^7 & -\omega^4 & \omega^8 & \omega^5 & \omega^5 & \omega^8 & \omega^{14}
\end{pmatrix},
\tag{37}
$$

where $\omega = e^{2\pi i/15}$ is now a *fifteenth* root of unity. Since all matrix elements can be written in terms of 30th roots of unity, S_{15} is an example of a Butson-Hadamard matrix $BH(15, 30)$. The vanishing defect of this matrix, i.e. $d(S_{15}) = 0$, implies that S_{15} is *isolated*. This property excludes S_{15} from being a member of either of the affine families $F_{15}^{(8)}$ or $(F_{15}^{(8)})^T$.

Proposition 4. *The matrix S_{15} is inequivalent to $F_{15}^{(8)}$ and $(F_{15}^{(8)})^T$.*

One could produce additional complex Hadamard matrices by choosing different combinations of MU bases from the complete set of six, such as $K_1 = I_5$ or $L_1 = F_3$. It is likely that various inequivalent matrices will result from these choices.

5 Examples: dimensions $d > 15$, and further generalisations

The construction of the matrices S_6, S_9 and S_{15} has a common feature: in each case, the matrices $K_0, \ldots, K_{p-1}, L_0, \ldots, L_{p-1}$ used to construct the blocks of the Hadamard matrix in Theorem 3 include a *complete* set of $(q + 1)$ MU bases of the space $\mathbb{C}^q$. For the cases $d = 6$ and $d = 15$, i.e. $q = 2p - 1$, complete sets of MU bases in dimension three and five

are used, respectively, resulting in the isolated matrices S_6 and S_{15}. Furthermore, in the case $d = 9$, where $q < 2p - 1$, a complete set of four MU bases in dimension three is used, and again we find an isolated Hadamard matrix, namely S_9.

Thus, one might expect additional isolated complex Hadamard matrices to emerge for larger composite dimensions whenever its factors are related by $q \leq 2p - 1$. We have been able to confirm this property for all primes p, q, with $pq < 100$ and $q \leq 2p - 1$, excluding the case $d = 4$. The first three examples are covered by S_6, S_9 and S_{15} which we already know are isolated. The remaining five matrices S_{25}, S_{35}, S_{49}, S_{77} and S_{91}, also turn out to be *isolated*. We construct these matrices as follows:

- S_{25} is derived from the complete set of six MU bases I_5, F_5 and $H_j = D^j F_5$, $j = 1 \ldots 4$, where $D = \mathrm{diag}(1, \omega, \omega^4, \omega^4, \omega)$ and $\omega = e^{2\pi i/5}$. The matrices K_n, for $n = 0 \ldots 4$, are chosen as I_5, I_5, I_5, H_1, H_2, respectively, and L_n as F_5, F_5, F_5, H_3, H_4, respectively.

- S_{35} uses the complete set of eight MU bases I_7, F_7 and $H_j = D^j F_7$, $j = 1 \ldots 6$, where $D = \mathrm{diag}(1, 1, \omega, \omega^3, \omega^6, \omega^3, \omega)$ and $\omega = e^{2\pi i/7}$. The matrices K_n, for $n = 0 \ldots 4$, are chosen as I_7, I_7, H_1, H_2, H_3, and L_n as F_7, F_7, H_4, H_5, H_6, respectively.

- S_{49} is constructed from the same complete set of MU bases used for S_{35}, and we choose K_n as $I_7, I_7, I_7, I_7, H_1, H_2, H_3$, and L_n as $F_7, F_7, F_7, F_7, H_3, H_4, H_5$, with $n = 0 \ldots 6$, respectively.

- S_{77} uses a complete set of twelve MU bases given by I_{11}, F_{11} and H_j, $j = 1 \ldots 10$ with $D = \mathrm{diag}(1, 1, \omega, \omega^3, \omega^6, \omega^{10}, \omega^4, \omega^{10}, \omega^6, \omega^3, \omega)$ and $\omega = e^{2\pi i/11}$. The matrices K_n, for $n = 0 \ldots 6$, are chosen as $I_{11}, I_{11}, H_1, H_2, H_3, H_4, H_5$ and L_n as $F_{11}, F_{11}, H_6, H_7, H_8, H_9, H_{10}$, respectively.

- S_{91} is based on the complete set of fourteen MU bases in $\mathbb{C}^{13}$, i.e. the identity I_{13}, the Fourier matrix F_{13}, and the matrices $H_j = D^j F_{13}$ for $j = 1 \ldots 12$, where the diagonal matrix is given by $D = \mathrm{diag}(1, 1, \omega, \omega^3, \omega^6, \omega^{10}, \omega^2, \omega^8, \omega^2, \omega^{10}, \omega^6, \omega^3, \omega)$ and $\omega = e^{2\pi i/13}$. The matrices $K_1, \ldots, K_6, L_1, \ldots, L_6$, correspond to $H_1, \ldots, H_{12}$, respectively.

All these isolated matrices are of Butson-Hadamard type with $S_{25} \in BH(25, 10)$, $S_{35} \in BH(35, 70)$, $S_{49} \in BH(49, 14)$, $S_{77} \in BH(77, 154)$ and $S_{91} \in BH(91, 182)$. They may have smaller roots of unity if their matrix elements contain no entries equal to (-1). The matrix S_{91} is similar to S_6 and S_{15} in the sense that the prime factors of $d = 91$ satisfy the equality $q = 2p - 1$, meaning that each MU basis from the complete set is used exactly once. For the other isolated matrices, the factors satisfy the inequality $q < 2p - 1$, which implies that some MU bases are used more than once in the set $K_0, \ldots, L_{p-1}$. In this case, there are additional choices for the bases used; different combinations may lead to further inequivalent isolated complex Hadamard matrices.

So far, we have applied Theorem 3 mainly to product dimensions $4 \leq d \leq 15$ or when $d = pq < 100$ and $q \leq 2p - 1$. To explore whether the latter constraint on the factors p and q is necessary, we have constructed Hadamard matrices in all other composite dimension

$d < 100$ for $p, q \leq 13$. In each of these cases, i.e. $d = 21, 22, 26, 33, 39, 55$ and 65, we were able to identify isolated Hadamard matrices. In addition, it is also possible to construct Hadamard matrices with non-zero defects, simply by selecting different sets of MU bases for $K_0, \ldots, L_{p-1}$. Thus, the theorem is potentially the source of infinitely many new Hadamard matrices in *arbitrary* product dimensions.

Interestingly, the method is not limited to dimensions of the form $d = pq$: if the numbers p and q are composite, it is likely that additional, possibly inequivalent complex Hadamard matrices can be constructed which relate to different factorisations of the dimension, such as $2 \times 6 = 3 \times 4$ when $d = 12$. Furthermore, it has been shown [40] that *inequivalent* complete sets of MU bases exist for large prime powers $d = p^n$, possibly leading to yet more inequivalent Hadamard matrices. One could also try to create continuous families of complex Hadamard matrices if sets of four or more MU bases exist which contain free parameters after dephasing.

Finally, another generalisation of Theorem 3 can be achieved as follows. The Hadamard matrices we have constructed are derived from product bases which tensor each vector in an orthonormal basis of $\mathbb{C}^p$ with an orthonormal basis of $\mathbb{C}^q$ (cf. Eqs. (7,8)). However, other types of product bases exist; for example, one could take vectors from *different* orthonormal bases in $\mathbb{C}^p$ and tensor them with vectors from *one* basis in $\mathbb{C}^q$. The classification of all product bases in the space $\mathbb{C}^2 \otimes \mathbb{C}^3$, up to local equivalence transformations, contains a number of examples of these so-called *indirect* product bases [24]. Thus, alternative block structures may be allowed in Theorem 3, potentially leading to other Hadamard matrices.

6 Summary and outlook

The main results of this paper are (i) a new general construction of complex Hadamard matrices in composite dimensions $d = pq$ (p, q prime) described in Theorem 3, and (ii) the explicit derivation of various new complex Hadamard matrices as a consequence of this theorem. The construction relies on the simple idea that a suitable unitary transformation maps a pair of MU *product* bases to its standard form in which the vectors of one basis turn into the columns of a complex Hadamard matrix. It becomes possible to *systematically* construct new Hadamard matrices many of which are isolated. Previous examples of isolated Hadamard matrices have been found by trial and error [13] or from numerical methods [34].

To illustrate the approach we first derive some known results in low dimensions. In particular, we find the complete family of complex Hadamard matrices when $d = 4$, and in dimension six we find the isolated matrix S_6. We then proceed to higher dimensions, obtaining isolated Hadamard matrices of order 9, 10, 14 and 15. Two of these are new isolated Butson-type Hadamard matrices, namely $S_9 \in BH(9, 6)$ and $S_{15} \in BH(15, 30)$, while $S_{10} \in BH(10, 5)$ and $S_{14} \in B(14, 7)$ are shown to be inequivalent to nearly all known

Hadamard matrices of their order. However, we cannot exclude the equivalence of S_{10} to B_{10} or X_{10}, and of S_{14} to B_{14}.

In dimensions $d = 10$ and $d = 14$, there is some flexibility in selecting suitable subsets of MU bases when applying Theorem 3. This enables us to construct two non-isolated Hadamard matrices S'_{10} and S'_{14}, with defects equal to 8 and 12, respectively. Further research is needed to understand which choices of MU bases will lead to inequivalent Hadamard matrices.

Whenever the factors in the product dimension $d = pq$ are related by $q \leq 2p - 1$, the set $K_0, \ldots, K_{p-1}, L_0, \ldots, L_{p-1}$ given in Theorem 3 can accommodate a *complete* set of MU bases for the space $\mathbb{C}^q$. We speculate that in these cases, with the exception of dimension four, Theorem 3 will always give rise to an isolated Hadamard matrix. This expectation has been confirmed for all matrices of order $d = pq < 100$ that satisfy $q \leq 2p - 1$. In these cases, the matrices S_6, S_9, and S_{15}, as well as S_{25}, S_{35}, S_{49}, S_{77} and S_{91} all turn out to be isolated, and they include the largest known examples of isolated Hadamard matrices (as far as we know). What is more, we are also able to generate isolated and (non-isolated) Hadamard matrices for dimensions $d = 21, 22, 26, 33, 39, 55$ and 65, giving rise to a total of 16 isolated complex Hadamard matrices. Twelve of them are new, while the remaining four, namely $S_{10}, S_{14}, S_{22}, S_{26} \in BH(2p, p)$, may be equivalent to matrices resulting from Butson's construction.

Throughout this paper we have limited our search to Butson-type Hadamard matrices. However, the method given in Theorem 3 covers a much wider class of complex Hadamard matrices. We expect that many other examples of more general Hadamard matrices can be found by extending the choice for the unitary matrices K_n and L_n.

Acknowledgments

A   Appendix: Explicit construction of $BH(10,5)$ and $BH(14,7)$

In this Appendix we list the two Butson-type Hadamard matrices $BH(2p,p)$ of order 10 and 14, which we derive from the construction given in Butson's original paper [33].

• For $p = 5$, the dephased matrix $BH(10,5)$ is given by

$$
B_{10} = \frac{1}{\sqrt{10}}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^2 & \omega^4 & \omega & \omega^3 & 1 \\
1 & \omega^2 & \omega^4 & \omega & \omega^3 & \omega^3 & \omega^2 & \omega & 1 & \omega^4 \\
1 & \omega^3 & \omega & \omega^4 & \omega^2 & \omega^3 & \omega^4 & 1 & \omega & \omega^2 \\
1 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^2 & 1 & \omega^3 & \omega & \omega^4 \\
1 & \omega^3 & \omega^2 & \omega^2 & \omega^3 & 1 & \omega & \omega^4 & \omega^4 & \omega \\
1 & \omega^2 & 1 & \omega^4 & \omega^4 & \omega & \omega & \omega^3 & \omega^2 & \omega^3 \\
1 & \omega & \omega^3 & \omega & 1 & \omega^4 & \omega^3 & \omega^4 & \omega^2 & \omega^2 \\
1 & 1 & \omega & \omega^3 & \omega & \omega^4 & \omega^2 & \omega^2 & \omega^4 & \omega^3 \\
1 & \omega^4 & \omega^4 & 1 & \omega^2 & \omega & \omega^3 & \omega^2 & \omega^3 & \omega
\end{pmatrix},
\tag{38}
$$

where $\omega = e^{2\pi i/5}$ is a fifth root of unity. The defect of B_{10} is zero.

• For $p = 7$, the dephased matrix $BH(14,7)$ is

$$
B_{14} = \frac{1}{\sqrt{14}}
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & 1 & \omega^3 \\
1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 & \omega^5 & \omega^3 & \omega^2 & \omega & 1 & \omega^6 & \omega^5 & \omega^4 \\
1 & \omega^3 & \omega^6 & \omega^2 & \omega^5 & \omega & \omega^4 & \omega^5 & 1 & \omega^2 & \omega^4 & \omega^6 & \omega & \omega^3 \\
1 & \omega^4 & \omega & \omega^5 & \omega^2 & \omega^6 & \omega^3 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & 1 \\
1 & \omega^5 & \omega^3 & \omega & \omega^6 & \omega^4 & \omega^2 & \omega^3 & \omega^4 & \omega^5 & \omega^6 & 1 & \omega & \omega^2 \\
1 & \omega^6 & \omega^5 & \omega^4 & \omega^3 & \omega^2 & \omega & \omega^6 & \omega^3 & 1 & \omega^4 & \omega & \omega^5 & \omega^2 \\
1 & \omega^4 & \omega^2 & \omega & \omega & \omega^2 & \omega^4 & 1 & \omega^5 & \omega^6 & \omega^3 & \omega^3 & \omega^6 & \omega^5 \\
1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & 1 & \omega^4 & \omega^6 & \omega^4 & \omega^5 & \omega^2 & \omega^2 & \omega^5 \\
1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & 1 & \omega^3 & \omega^2 & \omega & \omega^3 & \omega & \omega^2 & \omega^6 & \omega^6 \\
1 & \omega^2 & \omega^5 & \omega^2 & 1 & \omega^6 & \omega^6 & \omega & \omega^4 & \omega^3 & \omega^5 & \omega^3 & \omega^4 & \omega \\
1 & \omega^6 & \omega^6 & 1 & \omega^2 & \omega^5 & \omega^2 & \omega & \omega & \omega^4 & \omega^3 & \omega^5 & \omega^3 & \omega^4 \\
1 & \omega^3 & 1 & \omega^5 & \omega^4 & \omega^4 & \omega^5 & \omega^2 & \omega^6 & \omega^6 & \omega^2 & \omega & \omega^3 & \omega \\
1 & 1 & \omega & \omega^3 & \omega^6 & \omega^3 & \omega & \omega^4 & \omega^5 & \omega^2 & \omega^2 & \omega^5 & \omega^4 & \omega^6
\end{pmatrix},
\tag{39}
$$

with $\omega = e^{2\pi i/7}$ being a seventh root of unity. This matrix has zero defect.

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