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RESEARCH ARTICLE

On quasi-orthogonal cocycles

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Abstract

We introduce the notion of *quasi-orthogonal cocycle*. This is motivated in part by the maximal determinant problem for square $\{\pm 1\}$-matrices of size congruent to 2 modulo 4. Quasi-orthogonal cocycles are analogous to the *orthogonal cocycles* of algebraic design theory. Equivalences with new and known combinatorial objects afforded by this analogy, such as quasi-Hadamard groups, relative quasi-difference sets, and certain partially balanced incomplete block designs, are proved.

KEYWORDS

cocycle, (quasi-)orthogonal, block design, difference set, (quasi-) Hadamard group

1 | INTRODUCTION

In the early 1990s, de Launey and Horadam discovered *cocyclic development of pairwise combinatorial designs*. Their discovery opened up a new area in design theory, emphasizing algebraic methods drawn mainly from group theory and cohomology. See [7,12] for comprehensive expositions.

Let G and U be finite groups, with U abelian. A map $\psi : G \times G \to U$ such that

$$\psi(g,h)\psi(gh,k) = \psi(g,hk)\psi(h,k) \quad \forall\, g,h,k \in G \tag{1}$$

is a *cocycle* (*over G, with coefficients in U*). We may assume that ψ is normalized, i.e. $\psi(1,1) = 1$. For any (normalized) map $\phi : G \to U$, the cocycle $\partial\phi$ defined by $\partial\phi(g,h) = \phi(g)^{-1}\phi(h)^{-1}\phi(gh)$ is a *coboundary*. The set of cocycles $\psi : G \times G \to U$ forms an abelian group $Z^2(G,U)$ under pointwise multiplication. The quotient of $Z^2(G,U)$ by the subgroup of coboundaries is the *second cohomology group of G with coefficients in U*, denoted $H^2(G,U)$.

Each cocycle $\psi \in Z^2(G,U)$ is displayed as a *cocyclic matrix* M_ψ: under some indexings of the rows and columns by G, M_ψ has entry $\psi(g,h)$ in position (g,h). Our principal focus in this paper is the case $U = \langle -1 \rangle \cong \mathbb{Z}_2$. We say that ψ is *orthogonal* if M_ψ is a Hadamard matrix, i.e. $M_\psi M_\psi^\top = nI_n$ where $n = |G|$ is necessarily 1, 2, or a multiple of 4.

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The paper [6] describes explicit links between orthogonal cocycles and other combinatorial objects. For example, we can use an orthogonal cocycle to construct a relative difference set with forbidden subgroup $\mathbb{Z}_2$ in a central extension of $\mathbb{Z}_2$ by G, and vice versa. Such extensions, known as *Hadamard groups*, were studied by Ito in a series of papers beginning with [13]. Their equivalence with cocyclic Hadamard matrices was demonstrated in [8]. They are further equivalent to class regular group divisible designs on which the Hadamard group acts as a regular group of automorphisms. Techniques and results have been translated fruitfully between the different contexts.

Recently, cocycles over groups G of even order not divisible by 4 have been examined as a source of $(-1, 1)$-matrices with maximal determinant [1,3]. In this paper, we discuss existence, classification, and combinatorics of such cocycles under the appropriate version of orthogonality—modifying a familiar balance condition on rows (and columns) of the cocyclic matrix when $|G|$ is divisible by 4. In particular, we prove versions of the equivalences in [6]. The paper is a launching point for investigation of all these new algebraic and combinatorial ideas.

Throughout, I denotes an identity matrix and J a square all-1s matrix. The Kronecker product of $A = [a_{i,j}]$ and B is $A \otimes B := [a_{i,j}B]$. Given a matrix $M = [m_{i,j}]$, we write $\mathrm{abs}(M)$ for $[|m_{i,j}|]$.

2 | QUASI-ORTHOGONAL COCYCLES

A Hadamard matrix with normalized first row (each entry equal to 1) has zero row sums everywhere else. The same statement with "row" replaced by "column" is also true. As it happens, this constraint on rows and columns characterizes the cocyclic matrices that are Hadamard: $\psi \in Z^2(G, \mathbb{Z}_2)$ is orthogonal if and only if $|\{h \in G \mid \psi(g, h) = 1\}| = |G|/2$ (equivalently, $|\{h \in G \mid \psi(h, g) = 1\}| = |G|/2$) for each $g \in G \setminus \{1\}$.

Let $M = [m_{i,j}]$ be an $n \times n$ $(-1, 1)$-matrix with normalized first row. The *row excess*

$$RE(M) = \sum_{i=2}^{n} \left| \sum_{j=1}^{n} m_{i,j} \right|$$

measures how close the row sums of M are to zero. Assuming that $n \equiv 0 \mod 4$, a cocycle ψ over a group G of order n is orthogonal if and only if $RE(M_\psi) = 0$. We will give an appropriate minimality condition on row excess for cocyclic matrices of orders $n \equiv 2 \mod 4$.

Denote the Grammian $MM^\top$ by $\mathrm{Gr}(M)$. Fix an ordering $g_1 = 1, g_2, \ldots, g_n$ of G to index $M_\psi = [\psi(g_i, g_j)]$. Manipulations with the cocycle identity (1) yield

Lemma 2.1 (Lemma 6.6 of [12]). $\mathrm{Gr}(M_\psi)$ *has* (i, j)*th entry*

$$\psi\left(g_i g_j^{-1}, g_j\right) \sum_{g \in G} \psi\left(g_i g_j^{-1}, g\right).$$

Unless stated otherwise, henceforth G is a group of order $4t + 2 \geq 6$. Thus G has a (normal) splitting subgroup of order $2t + 1$.

Each row of a $(-1, 1)$-matrix may be designated as even or odd, according to the parity of the number of 1s that it contains. Note that rows of different parity cannot occur in a Hadamard matrix of order > 2.

Proposition 2.2 (cf. Proposition 2 of [1]). *Let M be a cocyclic matrix with indexing group G and let e be the number of its even rows. Then*

(i) *$e = 4t + 2$ or $2t + 1$; so $RE(M) \geq 4t$.*

(ii) *$RE(M) = 4t$ if and only if*

$$\mathrm{abs}(\mathrm{Gr}(M)) = \begin{bmatrix} 4tI + 2J & 0 \\ 0 & 4tI + 2J \end{bmatrix} \tag{2}$$

up to row permutation.

Proof. Two rows of different (respectively, the same) parity in M have inner product 0 (respectively, 2) modulo 4. Hence $2e(4t + 2 - e)$ entries of $\mathrm{Gr}(M)$ are congruent to 0 modulo 4. On the other hand, because a row of M sums to 0 modulo 4 if and only if it is odd, Lemma 2.1 implies that each row of $\mathrm{Gr}(M)$ has precisely $4t + 2 - e$ entries congruent to 0 modulo 4. Now (i) is apparent.

If $RE(M) = 4t$ then we get the Grammian (2) after permuting rows of M so that the first $2t + 1$ rows are even. Conversely, if (2) holds then $e = 2t + 1$, the only noninitial rows of M with nonzero sum are rows $2, \ldots, 2t + 1$, and that sum is ± 2. □

Combined with our earlier observation that full orthogonality of a cocycle ψ is the same as $RE(M_\psi)$ being minimal, Proposition 2.2 suggests the following.

Definition 2.3. $\psi \in Z^2(G, \mathbb{Z}_2)$ is *quasi-orthogonal* if $RE(M_\psi) = 4t$.

The next result, a useful working characterization of quasi-orthogonality, essentially just rephrases Proposition 2.2 (ii).

Lemma 2.4. *For $\psi \in Z^2(G, \mathbb{Z}_2)$, let*

$$X_1 = \Big\{ g \in G \setminus \{1\} \Big| \sum_{h \in G} \psi(g, h) = \pm 2 \Big\}$$

and

$$X_2 = \Big\{ g \in G \setminus \{1\} \Big| \sum_{h \in G} \psi(g, h) = 0 \Big\}.$$

Then ψ is quasi-orthogonal if and only if $|X_1| = 2t$ and $|X_2| = 2t + 1$.

We record some facts about the existence of quasi-orthogonal cocycles.

Proposition 2.5. *No coboundary is quasi-orthogonal.*

Proof. Observe that $M = M_{\partial\phi}$ is Hadamard equivalent to the group-developed matrix $N = [\phi(gh)]_{gh}$. Thus, if $\partial\phi$ is quasi-orthogonal and $\mathrm{abs}(\mathrm{Gr}(M))$ has the form (2), then $\mathrm{abs}(\mathrm{Gr}(N))$ does as well. It follows that $J\mathrm{Gr}(N) \equiv 2J \mod 4$. Also $J\mathrm{Gr}(N) = k^2 J$ where k denotes the constant row and column sum of N. But of course $k^2 \not\equiv 2 \mod 4$. □

Remark 1. Indeed, every row of $M_{\partial\phi}$ is even; from which it is immediate that $\partial\phi$ cannot be quasi-orthogonal.

Remark 2. Orthogonal coboundaries exist (in square orders).

After carrying out exhaustive searches using MAGMA [4], we found quasi-orthogonal cocycles over every group of order $4t + 2 \leq 42$.

Example (R. Egan). Take any Hadamard matrix with circulant core and let A be the normalized core. Then $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \otimes A$ displays a quasi-orthogonal cocycle.

By contrast, groups over which there are no cocyclic Hadamard matrices start appearing at order 8. Also, from order 24 onwards there exist Hadamard matrices that are not cocyclic: see [16, Table 1].

A cocyclic matrix of order $4t+2$ whose determinant has absolute value attaining the Ehlich–Wojtas bound $2(4t+1)(4t)^{2t}$ must be quasi-orthogonal [1, Proposition 3]. Examples of quasi-orthogonal cocycles are thereby available in [1,2]. So far, we have not found a group G of order $4t+2$ such that $4t+1$ is the sum of two squares and there is no quasi-orthogonal cocycle over G whose matrix attains the Ehlich–Wojtas bound.

Cohomological equivalence of cocycles does not preserve orthogonality nor quasi-orthogonality. However, both properties are preserved by a certain "shift action" on each cocycle class. For $a \in G$, this action maps $\psi \in Z^2(G, \mathbb{Z}_2)$ to $\psi a := \psi \partial \psi_a$, where $\psi_a(x) = \psi(a, x)$; see [11, Definition 3.3]. By Lemma 2.1, the sum $\sum_{h \in G} \psi(a, h)\psi(ag, h)$ of row $g \neq 1$ in $M_{\psi a}$ is either a noninitial row sum of M_ψ, or the negation of one. Hence, by Lemma 2.4, ψa is quasi-orthogonal if and only if ψ is too (this is the same argument as the one in the proof of [11, Lemma 4.9] for orthogonal cocycles).

3 | QUASI-HADAMARD GROUPS

A group E of order $8t$ is a *Hadamard group* if it contains a *Hadamard subset*: a transversal T for the cosets of a central subgroup $Z \cong \mathbb{Z}_2$ such that $|T \cap xT| = 2t$ for all $x \in E \setminus Z$ (in fact $x \in T \setminus Z$ suffices; cf. Remark 3 below). These definitions are due to Ito [13]. He showed that the dicyclic group

$$Q_{8t} = \langle a, b \mid a^{2t} = b^2,\ b^4 = 1,\ b^{-1}ab = a^{-1} \rangle$$

is a Hadamard group whenever $2t-1$ or $4t-1$ is a prime power [14], and conjectured that Q_{8t} is always a Hadamard group. In [8], Hadamard groups are shown to coincide with cocyclic Hadamard matrices, and Ito's conjecture is verified for $t \leq 11$. Schmidt [18] later extended the verification up to $t = 46$.

We now define the analog of Hadamard group.

Definition 3.1. Let E be a group of order $8t+4 \geq 12$ with central subgroup $Z \cong \mathbb{Z}_2$. We say that E is a *quasi-Hadamard group* if there exists a transversal T for Z in E containing a subset $S \subset T \setminus Z$ of size $2t+1$ such that

$$|T \cap xT| = \begin{cases} 2t+1 & x \in S \\ 2t \text{ or } 2t+2 & x \in T \setminus (S \cup Z). \end{cases} \tag{3}$$

Remark 3. For any $x \in E$ and the nontrivial element z of Z, $|T \cap xT| = n$ if and only if $|T \cap xzT| = 4t+2-n$.

We call the transversal T in Definition 3.1 a *quasi-Hadamard subset* of E. It may be assumed that $1 \in T$.

Given a group G and $\psi \in Z^2(G, \langle -1 \rangle)$, denote by E_ψ the canonical central extension of $\langle -1 \rangle$ by G; this has elements $\{(\pm 1, g) \mid g \in G\}$ and multiplication $(u, g)\,(v, h) = (uv\psi(g, h), gh)$. In the other direction, suppose that E is a finite group with normalized transversal T for a central subgroup $\langle -1 \rangle \cong \mathbb{Z}_2$. Put $G = E/\langle -1 \rangle$ and $\sigma(t\langle -1 \rangle) = t$ for $t \in T$. The map $\psi_T : G \times G \to \langle -1 \rangle$ defined by $\psi_T(g, h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$ is a cocycle; furthermore, $E_{\psi_T} \cong E$.

Theorem 3.2 (cf. Propositions 3.3 and 3.4 of [8]).

(i) *If ψ is quasi-orthogonal then $T = \{(1, g) \mid g \in G\}$ is a quasi-Hadamard subset of E_ψ.*

(ii) *If E has quasi-Hadamard subset T then ψ_T is quasi-orthogonal.*

Proof.

(i) For each $x = (u, g) \in E_\psi$, $|T \cap xT|$ counts the number of $h \in G$ such that $\psi(g, h) = u$. Hence

$$|T \cap xT| = \begin{cases} 2t & x \in \{1\} \times X_{1,-} \ \cup \ \{-1\} \times X_{1,+} \\ 2t+1 & x \in \{-1, 1\} \times X_2 \\ 2t+2 & x \in \{1\} \times X_{1,+} \ \cup \ \{-1\} \times X_{1,-} \end{cases}$$

where $X_{1,\pm} = \{g \in G \setminus \{1\} \mid \sum_{h \in G} \psi(g, h) = \pm 2\}$, and X_2, $X_1 = X_{1,+} \cup X_{1,-}$ are as in Lemma 2.4. So (3) holds with $S = \{1\} \times X_2$.

(ii) Let S be as in Definition 3.1. Since $\psi_T(g, h) = 1 \Leftrightarrow \sigma(g)\sigma(h) \in T$, the number of $h \in G$ such that $\psi_T(g, h) = 1$ for fixed $g \neq 1$ is $|T \cap \sigma(g)^{-1}T| = |\sigma(g)T \cap T|$, which equals $2t + 1$ if $\sigma(g) \in S$ and $2t$ or $2t + 2$ otherwise, by (3). Now this part follows from Lemma 2.4, with $X_1 = \{g \in G \setminus \{1\} \mid \sigma(g) \notin S\}$ and $X_2 = \{g \in G \mid \sigma(g) \in S\}$. □

Theorem 3.2 shows that quasi-orthogonal cocycle and quasi-Hadamard group are essentially the same concept.

Let D_{4t+2} denote the dihedral group of order $4t + 2$. If $\psi \in Z^2(D_{4t+2}, \mathbb{Z}_2)$ is not a coboundary then E_ψ is the group Q_{8t+4} with presentation

$$\langle a, b \mid a^{2t+1} = b^2, \ b^4 = 1, \ b^{-1}ab = a^{-1} \rangle.$$

Note that $Q_{8t+4} \cong C_{2t+1} \rtimes C_4$. We propose an analog of Ito's conjecture that the cocycle class in $H^2(D_{4t}, \mathbb{Z}_2)$ labeled $(A, B, K) = (1, -1, -1)$ in [8] always has orthogonal elements; equivalently, Q_{8t} is always a Hadamard group.

Conjecture 1. *Q_{8t+4} is a quasi-Hadamard group for all $t \geq 1$.*

Conjecture 1 has been verified up to $t = 10$, by our computer search for quasi-orthogonal cocycles. Actually, for fixed isomorphism type of G, there are very few possible isomorphism types of quasi-Hadamard groups arising from cocycles over G.

Lemma 3.3. *$H^2(G, \mathbb{Z}_2) \cong \mathbb{Z}_2$.*

Proof. First, $H_2(G) \cong H_2(N)$ where $N \leq G$ is a splitting subgroup of index 2 (see, e.g., [15, 2.2.6, p. 35]). Then $H^2(G, \mathbb{Z}_2) \cong \mathrm{Ext}(G/G', \mathbb{Z}_2) \cong \mathbb{Z}_2$ by the Universal Coefficient Theorem, because $|H_2(N)|$ is odd. □

Lemma 3.3 and Proposition 2.5 imply

Corollary 3.4. *For each $t \geq 1$ and fixed G, there are at most two non-isomorphic quasi-Hadamard groups arising from elements of $Z^2(G, \mathbb{Z}_2)$.*

Remark 4. For example, if G is cyclic or dihedral then a quasi-Hadamard group must be isomorphic to C_{8t+4} or Q_{8t+4}.

Remark 5. While all quasi-Hadamard groups are solvable, there exist non-solvable Hadamard groups.

Besides Conjecture 1, Ito proved two results about Hadamard groups that have had important consequences for the existence question in the theory of cocyclic Hadamard matrices; see [7, Corollaries 15.6.2 and 15.6.5, pp. 184–185]. We quote these for comparison with the less interesting situation for quasi-Hadamard groups (each of which has Sylow 2-subgroup C_4).

Theorem 3.5. *Suppose that H is a cocyclic Hadamard matrix of order greater than 2 over a group G with cyclic Sylow 2-subgroups. Then H is group-developed over G; i.e. the corresponding Hadamard group does not have cyclic Sylow 2-subgroups.*

Theorem 3.6. *No Hadamard group has a dihedral Sylow 2-subgroup.*

4 | RELATIVE QUASI-DIFFERENCE SETS

Let E be a group of order vm with normal subgroup N of order m. A *relative (v, m, k, λ)-difference set in E with forbidden subgroup N* is a k-subset R of a transversal for N in E, such that if $x \in E \setminus N$ then $x = r_1 r_2^{-1}$ for exactly λ pairs $r_1, r_2 \in R$. The last condition may be rewritten as

$$|R \cap xR| = \lambda \quad \forall\, x \in E \setminus N. \tag{4}$$

An important special case in which $k = v$ is the following.

Proposition 4.1 (Corollary 2.5 of [6]). *Let $|G| = 4t$. A cocycle $\psi \in Z^2(G, \langle -1 \rangle)$ is orthogonal if and only if $\{(1, g) \mid g \in G\}$ is a relative $(4t, 2, 4t, 2t)$-difference set in E_ψ with forbidden subgroup $\langle(-1, 1)\rangle$.*

In other words, a relative $(4t, 2, 4t, 2t)$-difference set is a Hadamard subset of a Hadamard group, and vice versa. However, when t is odd, Hiramine [10] proved that there are no relative $(2t, 2, 2t, t)$-difference sets. So we need an analog of relative difference set for quasi-Hadamard groups.

Definition 4.2. Let E a group of order $8t + 4$, and Z a normal (hence central) subgroup of order 2. A *relative* $(4t + 2, 2, 4t + 2, 2t + 1)$*-quasi-difference set* in E with forbidden subgroup Z is a transversal R for Z in E containing a subset $S \subset R \setminus \{1\}$ of size $2t + 1$such that, for all $x \in E \setminus Z$,

$$\begin{array}{ll} |R \cap xR| = 2t + 1 & x \in sZ \text{ for some } s \in S \\ |R \cap xR| = 2t \text{ or } 2t + 2 & \text{otherwise.} \end{array} \tag{5}$$

The familiar default assumption is that relative (quasi-) difference sets are normalized, i.e. contain 1.

Example. $R = \{1, a, a^2, b, ab, a^2 b\}$ is a relative $(6, 2, 6, 3)$-quasi-difference set in $E = \langle a, b \mid a^3 = b^2,\ b^4 = 1,\ a^b = a^5 \rangle \cong Q_{12}$ with forbidden subgroup $Z = \langle a^3 \rangle$.

It is clear from the definitions and Remark 3 that a relative $(4t + 2, 2, 4t + 2, 2t + 1)$-quasi-difference set in E is precisely a quasi-Hadamard subset of E. Together with Theorem 3.2, we then have

Proposition 4.3. *A cocycle $\psi \in Z^2(G, \langle -1 \rangle)$ is quasi-orthogonal if and only if $\{(1, g) \mid g \in G\}$ is a relative $(4t + 2, 2, 4t + 2, 2t + 1)$-quasi-difference set in E_ψ with forbidden subgroup $\langle(-1, 1)\rangle$.*

When ψ is a coboundary, Proposition 4.1 gives an equivalence between group-developed Hadamard matrices, Menon–Hadamard difference sets, and normal relative difference sets in $\mathbb{Z}_2 \times G$ with forbidden subgroup $\mathbb{Z}_2 \times \{1_G\}$; see [6, Theorem 2.6, Corollary 2.7]. This result has no counterpart in the context of Proposition 4.3, since quasi-orthogonal coboundaries do not exist.

Suppose now that k is not necessarily equal to v. The link between orthogonal cocycle and relative difference set may be broadened in several ways. As shown in [9], a relative (v, m, k, λ)-difference set in E with forbidden subgroup N is equivalent to a factor pair of N by $G \cong E/N$ that is (v, m, k, λ)-orthogonal. The factor pair consists of a *factor set* $\psi : G \times G \to N$ and a *coupling* that together determine E; it is (v, m, k, λ)-*orthogonal* with respect to a k-set $D \subseteq G$ if for each $x \in G \setminus \{1\}$ the sequence $\{\psi(x, y)\}_{y \in D \cap x^{-1}D}$ is a listing of each element of N exactly λ times (see [9] or [12, Section 7.2]). If $m = 2$ then the coupling is trivial and the set of factor pairs of N by G is just $Z^2(G, \mathbb{Z}_2)$. Moreover, an orthogonal cocycle is an orthogonal factor pair (with $k = v$ and $\lambda = v/2$). The same is not true for quasi-orthogonal cocycles.

Proposition 4.4. *There is no* $(6, 2, k, \lambda)$*-orthogonal factor pair for any* $k, \lambda > 0$*. Thus, none of the quasi-orthogonal cocycles over the groups of order* 6 *is an orthogonal factor pair.*

Proof. If a factor pair of $\mathbb{Z}_2$ by G is $(v, 2, k, \lambda)$-orthogonal with respect to D then D is an ordinary $(v, k, 2\lambda)$-difference set in G. But nontrivial $(6, k, \lambda)$-difference sets do not exist. □

5 | PARTIALLY BALANCED INCOMPLETE BLOCK DESIGNS

A relative (v, m, k, λ)-difference set in E with forbidden subgroup N is equivalent to a divisible (v, m, k, λ)-design that is class regular with respect to N and has E as a regular group of automorphisms (E acts regularly on the points and blocks, while N acts regularly on each of the v point classes); see [17, Theorem 1.1.11, p. 13]. We establish the analogous passage between relative quasi-difference sets and partially balanced incomplete block designs. A reference for the standard material in this section is [5, VI.1 and VI.42].

Let X be a v-set and $R_0, R_1, \ldots, R_m$ be nonempty subsets of $X \times X$, called *associate classes*. The class R_i is represented by an *associate* (incidence) *matrix*, i.e. a (0, 1)-matrix A_i indexed by X, with 1 in row x and column $y \Leftrightarrow (x, y) \in R_i$. The R_is comprise an *association scheme on* X if

1. $A_0 = I$
2. $\sum_{i=0}^{m} A_i = J$
3. for all i, $A_i^\top = A_i$
4. for all i, j such that $i \le j$, there are $p_{ij}^k \in \mathbb{N}$ such that $A_i A_j = \sum_k p_{ij}^k A_k$.

Given such an association scheme, a *partially balanced incomplete block design* PBIBD(m) with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$ based on X has b blocks, all of size k, each $x \in X$ occurs in exactly r blocks, and if $(x, y) \in R_i$ then x, y occur together in exactly λ_i blocks.

Theorem 5.1 (42.4, pp. 562–563 of [5]). *Let N be an incidence matrix of a* PBIBD(m) *with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$ corresponding to an association scheme with associate matrices $A_0, \ldots, A_m$. Then*

$$NN^\top = rI + \sum_{i=1}^{m} \lambda_i A_i \quad \textit{and} \quad JN = kJ. \tag{6}$$

Conversely, a $v \times b$ (0, 1)-matrix N such that (6) *holds for associate matrices A_i of an association scheme is an incidence matrix of a* PBIBD(m) *with parameters $v, b, r, k, \lambda_1, \ldots, \lambda_m$.*

We now embark on the construction of a specific PBIBD(4). Let M be any $(-1, 1)$-matrix satisfying (2) (so that if M is cocyclic then the underlying cocycle is quasi-orthogonal). Form the expanded matrix

$$\mathcal{E}_M = \begin{bmatrix} M & -M \\ -M & M \end{bmatrix}.$$

Put $A = \frac{1}{2}(J + M)$ and $\bar{A} = \frac{1}{2}(J - M)$; then the $(0, 1)$-version of $\mathcal{E}_M$ is

$$\Phi = \begin{bmatrix} A & \bar{A} \\ \bar{A} & A \end{bmatrix}. \tag{7}$$

Clearly

$$J\Phi = (4t + 2)J. \tag{8}$$

Next, we check that

$$AA^\top + \bar{A}\bar{A}^\top = (4t + 2)I + (2t + 2)\Delta_1 + 2t\Delta_2 + (2t + 1)\left((J_2 - I_2) \otimes J_{2t+1}\right),$$

$$\bar{A}A^\top + A\bar{A}^\top = 2t\Delta_1 + (2t + 2)\Delta_2 + (2t + 1)\left(J_2 - I_2\right) \otimes J_{2t+1}$$

where

$$\Delta_1 = (\mathrm{Gr}(M) + 2(I_2 \otimes J_{2t+1}) - (4t + 4)I)/4,$$

$$\Delta_2 = (2(I_2 \otimes J_{2t+1}) + 4tI - \mathrm{Gr}(M))/4.$$

Thus

$$\Phi\Phi^\top = (4t + 2)A_0 + (2t + 1)A_2 + (2t + 2)A_3 + 2tA_4 \tag{9}$$

where $A_0 = I_{8t+4}$, $A_2 = J_2 \otimes (J_2 - I_2) \otimes J_{2t+1}$, $A_3 = I_2 \otimes \Delta_1 + (J_2 - I_2) \otimes \Delta_2$, and $A_4 = I_2 \otimes \Delta_2 + (J_2 - I_2) \otimes \Delta_1$. Let $A_1 = (J_2 - I_2) \otimes I_{4t+2}$. Then

- $A_1^2 = A_0$, $A_1A_2 = A_2$, $A_1A_3 = A_4$, $A_1A_4 = A_3$.
- $A_2^2 = (4t + 2)(A_0 + A_1 + A_3 + A_4)$, $A_2A_3 = A_2A_4 = 2tA_2$.
- $A_3^2 = A_4^2 = 2tA_0 + (2t - 1)A_j$, $A_3A_4 = 2tA_1 + (2t - 1)A_{7-j}$ where $j \in \{3, 4\}$.

So requirement 4 in the definition of association scheme holds. Requirements 1–3 hold as well. Therefore

Lemma 5.2. *A_0, A_1, A_2, A_3, A_4 as above are the associate matrices of an association scheme.*

We now have our desired PBIBD.

Proposition 5.3. *The matrix Φ as defined in* (7) *for any M satisfying* (2) *is an incidence matrix of a* PBIBD(4) *with parameters $v = b = 8t + 4$, $r = k = 4t + 2$, $\lambda_1 = 0$, $\lambda_2 = 2t + 1$, $\lambda_3 = 2t + 2$, and $\lambda_4 = 2t$.*

Proof. This follows from (8), (9), Lemma 5.2, and Theorem 5.1. □

Example. Let $t = 1$ in Proposition 5.3. We choose a quasi-orthogonal cocycle over D_6 whose matrix A is visible in the top left quadrant of

$$\Phi = \begin{bmatrix} 1&1&1&1&1&1&0&0&0&0&0&0\\ 1&0&0&1&1&1&0&1&1&0&0&0\\ 1&0&1&0&0&0&0&1&0&1&1&1\\ 1&1&0&0&1&0&0&0&1&1&0&1\\ 1&1&0&0&0&1&0&0&1&1&1&0\\ 1&1&0&1&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&1&1&1&1&1&1\\ 0&1&1&0&0&0&1&0&0&1&1&1\\ 0&1&0&1&1&1&1&0&1&0&0&0\\ 0&0&1&1&0&1&1&1&0&0&1&0\\ 0&0&1&1&1&0&1&1&0&0&0&1\\ 0&0&1&0&1&1&1&1&0&1&0&0 \end{bmatrix}.$$

The nontrivial associate matrices are

$$A_1 = \begin{bmatrix} 0_6 & I_6 \\ I_6 & 0_6 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0_3 & J_3 & 0_3 & J_3 \\ J_3 & 0_3 & J_3 & 0_3 \\ 0_3 & J_3 & 0_3 & J_3 \\ J_3 & 0_3 & J_3 & 0_3 \end{bmatrix}, \; A_3 = \begin{bmatrix} \Delta_1 & \Delta_2 \\ \Delta_2 & \Delta_1 \end{bmatrix}, \; A_4 = \begin{bmatrix} \Delta_2 & \Delta_1 \\ \Delta_1 & \Delta_2 \end{bmatrix}$$

where

$$\Delta_1 = \begin{bmatrix} 0&1&0&0&0&0\\ 1&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&1&1\\ 0&0&0&1&0&1\\ 0&0&0&1&1&0 \end{bmatrix}, \quad \Delta_2 = \begin{bmatrix} 0&0&1&0&0&0\\ 0&0&1&0&0&0\\ 1&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0 \end{bmatrix}.$$

From now on, the notation R_i, A_i is reserved for the association scheme of Lemma 5.2, and Φ is an incidence matrix of a corresponding PBIBD(4) with parameters $v = b = 8t + 4$, $r = k = 4t + 2$, $\lambda_1 = 0$, $\lambda_2 = 2t + 1$, $\lambda_3 = 2t + 2$, $\lambda_4 = 2t$.

The next two theorems connect partially balanced incomplete block designs to quasi-orthogonal cocycles.

Theorem 5.4. *If $\psi \in Z^2(G, \langle -1 \rangle)$ is quasi-orthogonal then E_ψ is a regular group of automorphisms of the* PBIBD(4) *as in Proposition* 5.3. *The design is R_1-class regular with respect to $\langle (-1, 1) \rangle$.*

Proof. (Cf. [6, pp. 54–55].) Choose any ordering $1, g_2, \ldots, g_{4t+2}$ of G, and index $\mathcal{E}_{M_\psi}$ by $E = E_\psi$ under the ordering $(1, 1), \ldots, (1, g_{4t+2}), (-1, 1), \ldots, (-1, g_{4t+2})$. Then

$$\mathcal{E}_{M_\psi} = [\phi(xy)]_{x,y \in E}$$

where $\phi : (u, g) \mapsto u$. That is, $\mathcal{E}_{M_\psi}$ is group-developed over E. Hence E acts as a regular group of permutation automorphisms of $\mathcal{E}_{M_\psi}$; see [7, Theorem 10.3.8, pp. 123–124]. Each of the point classes $\{(1, g_i), (-1, g_i)\}$ prescribed by R_1 is stabilized by $\langle (-1, 1) \rangle$. □

Theorem 5.5. *Suppose that a* PBIBD(4) *with incidence matrix* Φ *has a central extension* E *of* $\langle -1 \rangle$ *as a regular group of automorphisms, and is* R_1*-class regular with respect to* $\langle -1 \rangle$. *Then there exists a relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set in* E *with forbidden subgroup* $\langle -1 \rangle$.

Proof. By [17, p. 15] and the hypothesis that E is regular, $\Phi^\top \Phi = \Phi\Phi^\top$. Thus $\Phi^\top$ is an incidence matrix for a PBIBD(4) with the same parameters as those of Φ. Index Φ by the elements $x_1 = 1, x_2, \ldots, x_{8t+4}$ of E, where x_i shifts column 1 to column i. Note that $x_{4t+2+i} = -x_i$ because Φ is R_1-class regular with respect to $\langle -1 \rangle$. Let $R = \{x \in E \mid \Phi_{1,x} = 1\}$. Since $\lambda_1 = 0$, R is a transversal for $\langle -1 \rangle$ in E. Also $x^{-1}R = \{y \in E \mid \Phi_{x,y} = 1\}$; then $|R \cap xR| = |R \cap x^{-1}R| = (\Phi\Phi^\top)_{1,x}$ for any $x \in E$. Inspection of the first row of $\Phi\Phi^\top$ reveals that R and $S = \{x \in E \mid (\Phi\Phi^\top)_{1,x} = 2t+1 \text{ and } \Phi_{1,x} = 1\}$ satisfy (5). □

Remark 6. Theorem 5.4 and $\Phi^\top\Phi = \Phi\Phi^\top$ imply that if ψ is quasi-orthogonal then $\mathrm{Gr}(M_\psi) = \mathrm{Gr}(M_\psi^\top)$. Definition 2.3 may therefore be framed equivalently in terms of column excess rather than row excess. (However, note that the transpose of a cocyclic matrix indexed by a non-abelian group need not even be Hadamard equivalent to a cocyclic matrix.)

Our final result should be compared with [6, Theorem 2.4] and [12, Corollary 7.31, p. 152].

Theorem 5.6. *The following are equivalent.*

I. $Z^2(G, \langle -1 \rangle)$ *contains a quasi-orthogonal cocycle.*

II. *There is a relative* $(4t+2, 2, 4t+2, 2t+1)$*-quasi-difference set with forbidden subgroup* $\langle -1 \rangle$ *in a quasi-Hadamard group* E *such that* $E/\langle -1 \rangle \cong G$.

III. *There exists a* PBIBD(4) *with incidence matrix* Φ *on which a quasi-Hadamard group* E *such that* $E/\langle -1 \rangle \cong G$ *acts regularly, and which is* R_1*-class regular with respect to* $\langle -1 \rangle$.

Proof. We have I $\Leftrightarrow$ II by Theorem 3.2 and Proposition 4.3, I $\Rightarrow$ III by Theorem 5.4, and III $\Rightarrow$ II by Theorem 5.5. □

Remark 7. The results cited in the proof of Theorem 5.6 enable us to explicitly construct each object from any other equivalent object.

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