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# Surface waves in deformed Bell materials

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## Abstract

Small amplitude inhomogeneous plane waves are studied as they propagate on the free surface of a predeformed semi-infinite body made of Bell constrained material. The predeformation corresponds to a finite static pure homogeneous strain. The surface wave propagates in a principal direction of strain and is attenuated in another principal direction, orthogonal to the free surface. For these waves, the secular equation giving the speed of propagation is established by the method of first integrals. This equation is not the same as the secular equation for incompressible half-spaces, even though the Bell constraint and the incompressibility constraint coincide in the isotropic infinitesimal limit. © 2002 Elsevier Science Ltd. All rights reserved.

## 1. Introduction

The so-called ‘Bell constraint’ was established empirically by Bell in the 1980s, by conducting a great body of experiments on certain metals [1]. Having subjected various annealed metals to different finite deformations, he concluded that the condition  $\text{tr } \mathbf{V} = 3$ , where  $\mathbf{V}$  is the left Cauchy–Green tensor, was always satisfied for his samples. Later, Beatty and Hayes [2] viewed this constraint as a purely kinematical one and used the theory of constrained finite elasticity to model the behaviour of the materials studied by Bell; it turned out that their theoretical predictions agreed remarkably with the experimental results (a detailed account of these advances, as well as references to Bell’s original articles, can be found in a recent review article by Beatty [3].) Hence a new branch of constrained finite elasticity was born and a substantial literature followed, on topics such as: finite homogeneous [2] and non-homogeneous [4] deformations, small deformations superimposed on large [5], universal relations

[6] and motions [7], non-linear fracture mechanics [8], stability analyses [9–12], Cauchy stress field analysis [13], etc. Small amplitude waves propagating in a finitely deformed ‘Bell material’ were examined in an unbounded media for the case of homogeneous plane waves [5,11] and in a long circular cylinder for the case of torsional waves [14]. The object of this paper is to establish the equation giving the speed of waves propagating on the plane free surface of a homogeneously deformed half-space made of Bell material (the secular equation).

Elastic surface waves on solid half-spaces were first studied by Rayleigh [15] in the context of linear isotropic elasticity. Subsequently, the theory of elastic surface waves evolved in a parallel manner in the fields of linear anisotropic elasticity [16,17] and of finite elasticity under initial stress [18,19]. Interestingly, although Rayleigh treated the case of incompressibility in his paper, surface waves propagating in internally constrained, triaxially prestressed, elastic materials received only tardy attention (see for instance Dowhaik and Ogden [20], Chadwick [21], or Rogerson [22].) Recently, Destrade proposed a method, inspired by Mozhaev [23], to derive the

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explicit secular equation for surface acoustic waves in monoclinic elastic crystals [24]. In that paper, it was claimed that taking the first integrals of the tractions rather than of the displacements (as in [23]) to obtain the secular equation was a procedure that could ‘easily accommodate internal constraints’. Here, this claim is validated by the consideration of the Bell constraint for small amplitude surface waves travelling in a principal direction of a homogeneously deformed half-space.

In Section 2, the incremental equations corresponding to a small amplitude displacement superposed on a large deformation [5] are recalled. These equations are specialized to the consideration of a surface wave in a deformed Bell material in Section 3, where the equations of motion and the boundary conditions are written for the tractions on planes parallel to the free surface. Next in Section 4, the secular equation is obtained from the vanishing of a  $2 \times 2$  determinant. This secular equation is compared to that obtained for incompressible *finitely* prestressed bodies [20] and it is found that one secular equation may not be deduced from the other. However, because a Bell material behaves in *infinitesimal* motions like an incompressible one [2], the secular equation written when the half-space is undeformed should coincide with that found by Rayleigh for incompressible isotropic materials — it is proved that such is indeed the case.

## 2. Small motion superposed on a large deformation

Beatty and Hayes established the equations governing the elastic behaviour of a Bell constrained material subjected to finite deformations [2,4], as well as those describing small deformations superimposed on large [14]. Here, their results are recalled in the context of small amplitude plane waves propagating in a homogeneously deformed Bell constrained half-space.

For a general hyperelastic Bell material, the strain energy density  $\Sigma$  depends only on  $I_2$  and  $I_3$ , the respective second and third invariants of the left Cauchy–Green tensor  $\mathbf{V}$ ,

$$\begin{aligned} \Sigma &= \Sigma(I_2, I_3), \text{ where } I_2 = [(\text{tr } \mathbf{V})^2 - \text{tr}(\mathbf{V}^2)]/2, \\ I_3 &= \det \mathbf{V}. \end{aligned} \quad (2.1)$$

Also, the Bell constraint imposes that  $I_1$ , the first invariant of  $\mathbf{V}$ , satisfy

$$I_1 = \text{tr } \mathbf{V} = 3 \quad (2.2)$$

for all deformations and at all times. Because of this constraint, a workless stress  $p\mathbf{V}$  is introduced in the stress–strain relationship, where the scalar  $p$  is to be determined from the equations of motion and boundary and initial conditions. Hence the constitutive equation relates the Cauchy stress tensor  $\mathbf{T}$  to the left Cauchy–Green tensor as [2]

$$\mathbf{T} = p\mathbf{V} + \omega_0 \mathbf{1} + \omega_2 \mathbf{V}^2. \quad (2.3)$$

Here, the material response functions  $\omega_0$  and  $\omega_2$  are defined by

$$\omega_0 = \frac{\partial \Sigma}{\partial I_3}, \quad \omega_2 = -\frac{1}{I_3} \frac{\partial \Sigma}{\partial I_2} \quad (2.4)$$

and these quantities should satisfy the Beatty–Hayes  $A$ -inequalities [2]

$$\omega_0(I_2, I_3) \leq 0, \quad \omega_2(I_2, I_3) > 0. \quad (2.5)$$

A solid half-space made of hyperelastic Bell material is now considered. In an isotropic undeformed reference state, the positions of its material particles are denoted by  $\mathbf{X}$ . Next, the half-space is subjected to a finite static pure homogeneous deformation, so that particles originally at  $\mathbf{X}$  have moved to  $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$ . Using the principal directions of strain as a rectangular Cartesian coordinate system, this deformation is expressed as  $x_\alpha = \lambda_\alpha X_\alpha$  ( $\alpha = 1, 2, 3$ , no sum), where the principal stretch ratios  $\lambda_1, \lambda_2, \lambda_3$  satisfy  $\lambda_1 + \lambda_2 + \lambda_3 = 3$ , according to (2.2). Also, for this deformation,

$$I_2 = \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1, \quad I_3 = \lambda_1 \lambda_2 \lambda_3. \quad (2.6)$$

The surface  $x_2 = 0$  is assumed to be free of tractions and the following Cauchy stress tensor  $\mathbf{T}_0$  satisfies the equations of equilibrium [2]:

$$\begin{aligned} (T_0)_{\alpha\alpha} &= p_0 \lambda_\alpha + \omega_0 + \lambda_\alpha^2 \omega_2, \quad \alpha = 1, 2, 3, \\ (T_0)_{\alpha\beta} &= 0, \quad \alpha \neq \beta, \end{aligned} \quad (2.7)$$

where  $\omega_0$  and  $\omega_2$  are evaluated at  $I_2, I_3$  given by (2.6), and

$$p_0 = -(\omega_0 + \lambda_2^2 \omega_2) / \lambda_2. \quad (2.8)$$

Next, the particle at  $\mathbf{x}$  undergoes an incremental displacement  $\varepsilon \mathbf{u}(\mathbf{x}, t)$  (with associated incremental constraint parameter  $\varepsilon p^*(\mathbf{x}, t)$ ), so that this further motion is of the form  $\mathbf{x} + \varepsilon \mathbf{u}(\mathbf{x}, t)$  (and the constraint parameter is of the form  $p_0 + \varepsilon p^*(\mathbf{x}, t)$ ), where  $\varepsilon$  is a small parameter. Throughout the rest of the paper, terms of order higher than  $\varepsilon$  are neglected. The corresponding incremental Cauchy stress  $\mathbf{T}^*$  is given by Beatty and Hayes [5]:

$$\begin{aligned} T_{11}^* &= \lambda_1 p^* + (\lambda_1 p_0 + C_{11}) \frac{\partial u_1}{\partial x_1} \\ &\quad + C_{12} \frac{\partial u_2}{\partial x_2} + C_{13} \frac{\partial u_3}{\partial x_3}, \\ T_{22}^* &= \lambda_2 p^* + C_{21} \frac{\partial u_1}{\partial x_1} + (\lambda_2 p_0 + C_{22}) \frac{\partial u_2}{\partial x_2} \\ &\quad + C_{32} \frac{\partial u_3}{\partial x_3}, \\ T_{33}^* &= \lambda_3 p^* + C_{31} \frac{\partial u_1}{\partial x_1} + C_{32} \frac{\partial u_2}{\partial x_2} \\ &\quad + (\lambda_3 p_0 + C_{33}) \frac{\partial u_3}{\partial x_3}, \\ T_{12}^* &= b_3 \left( \lambda_2^2 \frac{\partial u_1}{\partial x_2} + \lambda_1^2 \frac{\partial u_2}{\partial x_1} \right), \\ T_{23}^* &= b_1 \left( \lambda_3^2 \frac{\partial u_2}{\partial x_3} + \lambda_2^2 \frac{\partial u_3}{\partial x_2} \right), \\ T_{13}^* &= b_2 \left( \lambda_3^2 \frac{\partial u_1}{\partial x_3} + \lambda_1^2 \frac{\partial u_3}{\partial x_1} \right), \end{aligned} \quad (2.9)$$

with

$$\begin{aligned} C_{\alpha\beta} &= 2\lambda_\alpha^2 \delta_{\alpha\beta} \omega_2 - \lambda_\beta^2 (\omega_{02} + \lambda_\alpha^2 \omega_{22}) \\ &\quad + \lambda_1 \lambda_2 \lambda_3 (\omega_{03} + \lambda_\alpha^2 \omega_{23}), \\ b_\gamma &= \omega_2 + p_0 / (3 - \lambda_\gamma), \end{aligned} \quad (2.10)$$

where the derivatives  $\omega_{0\Gamma}$ ,  $\omega_{2\Gamma}$  ( $\Gamma = 2, 3$ ) of the material response functions  $\omega_0$ ,  $\omega_2$  are taken with respect to  $I_\Gamma$  and evaluated at  $I_2, I_3$  given by (2.6). The incremental Bell constraint yields

$$\lambda_1 \frac{\partial u_1}{\partial x_1} + \lambda_2 \frac{\partial u_2}{\partial x_2} + \lambda_3 \frac{\partial u_3}{\partial x_3} = 0, \quad (2.11)$$

and the components of the incremental traction on planes parallel to the free surface  $x_2 = 0$  are [5]

$$\begin{aligned} t_{12}^* &= T_{12}^* - (\lambda_1 p_0 + \omega_0 + \lambda_1^2 \omega_2) \frac{\partial u_2}{\partial x_1}, \\ t_{22}^* &= T_{22}^*, \\ t_{32}^* &= T_{32}^* - (\lambda_3 p_0 + \omega_0 + \lambda_3^2 \omega_2) \frac{\partial u_3}{\partial x_2}. \end{aligned} \quad (2.12)$$

Finally, the incremental equations of motion read

$$\frac{\partial T_{ij}^*}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \quad (2.13)$$

where  $\rho$  is the material density of the half-space, when it is maintained in the state of static homogeneous deformation.

### 3. Principal surface waves

Now the analysis of small amplitude motions in a deformed Bell half-space is specialized to the consideration of a plane wave propagating sinusoidally on the free surface  $x_2 = 0$ , in the  $x_1$ -direction, with attenuation in the  $x_2$ -direction. Without loss of generality, the component  $u_3$  is taken to be zero.

Hence, the wave may be modelled as

$$\begin{aligned} u_\Gamma(x_1, x_2, t) &= U_\Gamma(x_2) e^{ik(x_1 - vt)}, \\ p^*(x_1, x_2, t) &= kP(x_2) e^{ik(x_1 - vt)}, \end{aligned} \quad (3.1)$$

where  $\Gamma = 1, 2$ ,  $k$  is the real wave number,  $v$  is the real wave speed of propagation, and the specific dependence of the amplitudes  $U_\Gamma$  and  $P$  on  $x_2$  need not be specified [23]. Consequently, the incremental Bell constraint (2.11) now reduces to

$$i\lambda_1 U_1 + \lambda_2 U_2' = 0, \quad (3.2)$$

where, here and henceforward, the prime denotes differentiation with respect to  $kx_2$ . The other incremental expressions recalled in Section 2 are also simplified.

The incremental Cauchy stress components are now given by

$$T_{11}^* = k[\lambda_1 P + i(\lambda_1 p_0 + C_{11})U_1 + C_{12}U_2'],$$

$$T_{22}^* = k[\lambda_2 P + iC_{21}U_1 + (\lambda_2 p_0 + C_{22})U_2'],$$

$$\begin{aligned}
 T_{33}^* &= k[\lambda_3 P + iC_{31}U_1 + C_{32})U_2'], \\
 T_{12}^* &= kb_3(\lambda_2^2 U_1' + i\lambda_1^2 U_2), \\
 T_{32}^* &= T_{13}^* = 0.
 \end{aligned}
 \tag{3.3}$$

The third incremental equation of motion (2.13)<sub>3</sub> is automatically satisfied, while the two others reduce to

$$i T_{1\Gamma}^* + (T_{\Gamma 2}^*)' = -k\rho v^2 U_\Gamma, \quad (\Gamma = 1, 2). \tag{3.4}$$

Also, the expressions for the tractions  $t_{12}^*, t_{22}^*, t_{32}^*$  (2.12) have reduced to

$$\begin{aligned}
 t_{12}^* &= T_{12}^* - ik(\lambda_1 p_0 + \omega_0 + \lambda_1^2 \omega_2)U_2, \\
 t_{22}^* &= T_{22}^*, \quad t_{32}^* = T_{32}^* = 0.
 \end{aligned}
 \tag{3.5}$$

Now a system of four first-order differential equations can be written, for the non-zero displacements  $U_1, U_2$ , and for the quantities  $t_1, t_2$ , defined in terms of the tractions as

$$t_1 = k^{-1}t_{12}^*, \quad t_2 = k^{-1}t_{22}^*. \tag{3.6}$$

Taking Eqs. (2.8), (2.10), and (3.2) into account, this system can be written as

$$\begin{aligned}
 t_1' + i\lambda_1 \lambda_2^{-1} t_2 - (\lambda_1 \lambda_2^{-1} C - \rho v^2)U_1 &= 0, \\
 t_2' + it_1 - [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]U_2 &= 0, \\
 U_2' + i\lambda_1 \lambda_2^{-1} U_1 &= 0, \\
 b_3 \lambda_2^2 U_1' + ib_3 \lambda_2^2 U_2 - t_1 &= 0,
 \end{aligned}
 \tag{3.7}$$

where  $C$  is defined by

$$\begin{aligned}
 C &= \lambda_1^{-1} \lambda_2 C_{11} + \lambda_1 \lambda_2^{-1} C_{22} - C_{12} \\
 &\quad - C_{21} - 2\omega_0 - (\lambda_1^2 + \lambda_2^2)\omega_2,
 \end{aligned}
 \tag{3.8}$$

and the following expression for  $b_3 \lambda_2^2$ , obtained from (2.10) and (2.8), has also been used:

$$b_3 \lambda_2^2 = -(\omega_0 - \lambda_1 \lambda_2 \omega_2)/(1 + \lambda_1 \lambda_2^{-1}). \tag{3.9}$$

The system must be solved when subject to the following boundary conditions. First, the surface  $x_2 = 0$  remains free of traction, so that

$$t_{\Gamma 2}^*(0) = 0, \quad (\Gamma = 1, 2) \tag{3.10}$$

and second, the displacement must vanish at infinite distance from the free surface, so that

$$U_\Gamma(\infty) = 0, \quad P(\infty) = 0, \quad (\Gamma = 1, 2). \tag{3.11}$$

#### 4. Secular equation

Here the secular equation is derived by applying the method of first integrals to the system (3.7).

Proceed as follows. First, rewrite the boundary conditions (3.10)–(3.11) as

$$t_\Gamma(0) = t_\Gamma(\infty) = 0, \quad (\Gamma = 1, 2). \tag{4.1}$$

These last conditions, together with (3.7)<sub>1,2</sub>, imply that

$$t_\Gamma'(\infty) = 0, \quad (\Gamma = 1, 2). \tag{4.2}$$

Next, obtain  $U_\Gamma$  in terms of  $t_\Gamma, t_\Gamma'$  from (3.7)<sub>1,2</sub>, and  $U_\Gamma'$  in terms of  $U_\Gamma, t_\Gamma$  from (3.7)<sub>3,4</sub>, or equivalently, in terms of  $t_\Gamma, t_\Gamma'$ . Finally, substitute these expressions into the derivatives of Eqs. (3.7)<sub>1,2</sub> to get,

$$\begin{aligned}
 [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]t_1'' + i\beta_{12}t_2' \\
 - \frac{1}{b_3 \lambda_2^2}(\lambda_1 \lambda_2^{-1} C - \rho v^2)(b_3 \lambda_1^2 - \rho v^2)t_1 &= 0, \\
 (\lambda_1 \lambda_2^{-1} C - \rho v^2)t_2'' + i\beta_{12}t_1' \\
 - \lambda_1^2 \lambda_2^{-2} [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]t_2 &= 0,
 \end{aligned}
 \tag{4.3}$$

where (3.9) has been used, and  $\beta_{12}$  is defined by

$$\beta_{12} = \lambda_1 \lambda_2^{-1} [b_3(\lambda_1^2 - \lambda_2^2) + C] - (1 + \lambda_1 \lambda_2^{-1})\rho v^2. \tag{4.4}$$

The quantity  $\beta_{12}$  does not play any role in the secular equation, as is now seen. Multiply (4.3)<sub>1</sub> by  $t_1'$  and (4.3)<sub>2</sub> by  $t_2'$ , and integrate with respect to  $kx_2$  between  $x_2 = 0$  and  $x_2 = \infty$ , to obtain, using (4.1) and (4.2),

$$\begin{aligned}
 [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]t_1'(0)^2 - 2i\beta_{12} \int t_1' t_2' &= 0, \\
 (\lambda_1 \lambda_2^{-1} C - \rho v^2)t_2'(0)^2 - 2i\beta_{12} \int t_1' t_2' &= 0,
 \end{aligned}
 \tag{4.5}$$

and therefore,

$$\begin{aligned}
 [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]t_1'(0)^2 \\
 - (\lambda_1 \lambda_2^{-1} C - \rho v^2)t_2'(0)^2 &= 0.
 \end{aligned}
 \tag{4.6}$$

Similarly, multiply (4.3)<sub>1</sub> by  $[b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]t_1' + i\beta_{12}t_2$  and (4.3)<sub>2</sub> by  $(\lambda_1\lambda_2^{-1}C - \rho v^2)t_2' + i\beta_{12}t_1$ , and integrate with respect to  $kx_2$  between  $x_2=0$  and  $x_2=\infty$ , to obtain, using (4.1) and (4.2),

$$\begin{aligned}
 & [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]^2 t_1'(0)^2 - 2i\beta_{12} \frac{1}{b_3\lambda_2^2} \\
 & \quad \times (\lambda_1\lambda_2^{-1}C - \rho v^2)(b_3\lambda_1^2 - \rho v^2) \int t_1 t_2 = 0, \\
 & (\lambda_1\lambda_2^{-1}C - \rho v^2)^2 t_2'(0)^2 \\
 & \quad - 2i\beta_{12}\lambda_1^2\lambda_2^{-2}[b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2] \int t_1 t_2 = 0
 \end{aligned} \tag{4.7}$$

and therefore,

$$\begin{aligned}
 & [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]^3 \lambda_1^2 \lambda_2^{-2} t_1'(0)^2 \\
 & \quad - \frac{1}{b_3\lambda_2^2} (\lambda_1\lambda_2^{-1}C - \rho v^2)^3 (b_3\lambda_1^2 - \rho v^2) t_2'(0)^2 = 0.
 \end{aligned} \tag{4.8}$$

The two linear homogeneous equations (4.6) and (4.8) for the two unknowns  $t_1'(0)^2$  and  $t_2'(0)^2$  yield non-trivial solutions only when the corresponding determinant is zero. This condition provides the *secular equation for surface waves propagating in a principal direction of a deformed Bell material* as

$$\begin{aligned}
 & (b_3\lambda_1^2 - \rho v^2)(\lambda_1\lambda_2^{-1}C - \rho v^2)^2 \\
 & \quad = b_3\lambda_1^2 [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]^2.
 \end{aligned} \tag{4.9}$$

This equation is distinct from that obtained for deformed *incompressible* materials by Dowaiikh and Ogden [20], which reads as

$$\begin{aligned}
 & (\tilde{b}_3\lambda_1^2 - \rho v^2)(\tilde{C} - \rho v^2)^2 \\
 & \quad = \tilde{b}_3\lambda_2^2 [\tilde{b}_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]^2,
 \end{aligned} \tag{4.10}$$

where  $\tilde{b}_3$  and  $\tilde{C}$  are defined in terms of the stretch ratios  $\lambda_1, \lambda_2$  and the stored energy density  $W$  of the incompressible material as

$$\begin{aligned}
 & \tilde{b}_3 = \left( \lambda_1 \frac{\partial W}{\partial \lambda_1} - \lambda_2 \frac{\partial W}{\partial \lambda_2} \right) / (\lambda_1^2 - \lambda_2^2), \\
 & \tilde{C} = \lambda_1^2 \frac{\partial^2 W}{\partial \lambda_1^2} - 2\lambda_1\lambda_2 \frac{\partial^2 W}{\partial \lambda_1 \partial \lambda_2} + \lambda_2^2 \frac{\partial^2 W}{\partial \lambda_2^2} + 2\lambda_2 \frac{\partial W}{\partial \lambda_2}.
 \end{aligned} \tag{4.11}$$

Although Eqs. (4.9) and (4.10) share a similar left-hand side, their right-hand sides are different and could not have been a priori deduced one from another. In particular,  $b_3$  and  $\lambda_1\lambda_2^{-1}C$  can formally be transformed into  $\tilde{b}_3$  and  $\tilde{C}$ , respectively, to get from the left-hand side of (4.9) to the left-hand side of (4.10), but this transformation does not work for the right-hand sides of the equations, because the factor  $\lambda_1^2$  is replaced by  $\lambda_2^2$ . Hence, when the influence of the prestrain on the propagation of surface waves is studied [20], the behaviour of Bell materials is in contrast to that of incompressible materials, except in the special degenerate case of a bi-axial prestrain such that  $\lambda_1 = \lambda_2$ .

However, when the Bell material is *unstrained* for the problem at hand ( $\lambda_1 = \lambda_2 = \lambda_3 = 1$ ), only infinitesimal deformations (the surface wave) occur in the now isotropic half-space. In such a case, Beatty and Hayes [2] show that the Bell constraint is equivalent to the incompressibility constraint. Here, the quantities  $C, b_3$  are then given by

$$\lambda_1\lambda_2^{-1}C = C = 2(\omega_2 - \omega_0) = 4b_3, \tag{4.12}$$

and the secular Eq. (4.9) reduces to

$$\begin{aligned}
 & (1 - \xi)(4 - \xi)^2 = \xi^2 \quad \text{or} \\
 & \xi^3 - 8\xi^2 + 24\xi - 16 = 0,
 \end{aligned} \tag{4.13}$$

where  $\xi = 2\rho v^2/(\omega_2 - \omega_0)$ . This last equation is that established by Rayleigh [15] for surface waves in isotropic linearly elastic incompressible half-spaces.

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