RAYLEIGH WAVES AND SURFACE STABILITY FOR BELL MATERIALS IN COMPRESSION; COMPARISON WITH RUBBER

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Summary

The stability of a Bell-constrained half-space in compression is studied. To this end, the propagation of Rayleigh waves on the surface of the material when it is maintained in a static state of triaxial prestrain is considered. The prestrain is such that the free surface of the half-space is a principal plane of deformation. The exact secular equation is established for surface waves travelling in a principal direction of strain with attenuation along the principal direction normal to the free plane. As the half-space is put under increasing compressive loads, the speed of the wave eventually tends to zero and the bifurcation criterion, or stability equation, is reached.

Then the analysis is specialized to specific forms of strain-energy functions and prestrain, and comparisons are made with results previously obtained in the case of incompressible neo-Hookean or Mooney–Rivlin materials. It is found that these rubber-like incompressible materials may be compressed more than 'Bell empirical model' materials, but not as much as 'Bell simple hyperelastic' materials, before the critical stretches, solutions to the bifurcation criterion, are reached. In passing, some classes of incompressible materials which possess a relative-universal bifurcation criterion are presented.

1. Introduction

The works of Maurice Anthony Biot (1905–1985) cover a wide range of topics in mechanics and applied mathematics. Although much attention has been devoted to his contributions to the 'acoustics, elasticity, and thermodynamics of porous media' (**1**), his results in finite and incremental elasticity (**2**) were also far-reaching and are still relevant to many contemporary problems. For instance, he wrote a series of articles (summarized in his textbook (**2**)) on the surface and interfacial instability of elastic media under compression and his results found applications in rubber elasticity, viscoelasticity, folding of inhomogeneous/multilayered media, geological structures, etc. The idea underlying his resolution of these problems is the following: consider a medium at rest under a finite compression; superpose an incremental inhomogeneous static deformation whose amplitude vanishes away from the interface; show that the initial compression leads to an interface deflection which is infinite; conclude that this condition corresponds to interface bulking or instability. Biot also noted that the dynamical counterparts to surface and interface stability analyses were Rayleigh and Stoneley wave propagations, respectively.

This paper studies the propagation of Rayleigh waves on the surface of a compressed, internally

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constrained, hyperelastic half-space. The corresponding 'bulking' or 'bifurcation' criterion is derived by determining under which compressive loads the wave speed tends to zero in the secular equation. Biot often considered materials subject to incompressibility, an internal constraint which in nonlinear elasticity imposes that $\det \mathbf{V} = 1$ at all times, where $\mathbf{V}$ is the left stretch tensor. Here, the materials considered are subject to the constraint of Bell (**3**), $\operatorname{tr} \mathbf{V} = 3$. Both constraints are equivalent in infinitesimal linear elasticity (they reduce to: $\operatorname{tr} \mathbf{E} = 0$, where $\mathbf{E}$ is the infinitesimal strain tensor) but lead to quite different results at finite strains. In particular, the secular equation for Rayleigh surface waves cannot be deduced for Bell materials from the incompressible case. This equation was obtained in (**4**) as a cubic for the squared wave speed. However, this cubic corresponds to the rationalization of the exact secular equation and has spurious roots (**5**); accordingly, the corresponding relevant bifurcation criterion would have to be selected carefully. Here the *exact* secular equation is found, as well as the exact bifurcation criterion.

For incompressible materials, the Mooney–Rivlin form of the strain-energy function Σ_{MR} brings satisfactory correlation between theory and experiments for rubber-like materials; this function Σ_{MR} is linear with respect to I_1 and I_2, the first and second invariants of the left Cauchy–Green strain tensor $\mathbf{B} = \mathbf{V}^2$. For Bell constrained materials, the strain-energy density for 'simple hyperelastic Bell materials' (**3**) is linear with respect to i_2 and i_3, the second and third invariants of $\mathbf{V}$. Regarding experimental results, the strain-energy function for 'Bell empirical model' (**6**), $\Sigma_{\mathrm{BEM}} = \frac{2}{3}\beta_0[2(3 - i_2)]^{\frac{3}{4}}$ (where β_0 is a material constant) is reported as consistent with many trials on annealed metals such as aluminium, copper, or zinc. After the equations governing the problem have been written and solved in section 2 for a general form of the strain-energy function for a Bell-constrained half-space, the analysis is specialized in section 3 to the two specific forms of strain-energy functions presented above, and the results are compared with those obtained by Biot for rubber-like materials. It turns out that the maximal compressive load that can be applied to a half-space before the bifurcation criterion is reached is larger (smaller) for simple hyperelastic Bell materials (Bell empirical model) than for Mooney–Rivlin incompressible materials. Also, the bifurcation criterion is the same for every material within each class, and an infinity of strain-energy densities for which incompressible half-spaces admit such 'universal' bifurcation criteria is presented in section 3.4. Finally in section 4, the pertinence of the notion of (in)stability for finitely deformed hyperelastic materials is briefly reviewed and the general interest of the Bell constraint, as opposed to the constraint of incompressibility, is discussed.

2. Resolution of the problem in the general case

2.1 *Finite pure homogeneous triaxial pre-stretch*

Let $(O, x_1, x_2, x_3) \equiv (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ be a Cartesian rectangular coordinate system. Let the half-space $x_2 \geqslant 0$ be occupied by a hyperelastic Bell-constrained material, with strain-energy density Σ. This material is subject to the internal constraint that for any deformation,

$$i_1 \equiv \operatorname{tr} \mathbf{V} = 3 \tag{2.1}$$

at all times (**3**, **7**), where $\mathbf{V}$ is the left stretch tensor. Hence, for isotropic Bell materials, Σ depends only upon i_2 and i_3, the respective second and third invariants of $\mathbf{V}$. So, $\Sigma = \Sigma(i_2, i_3)$, where $i_2 = [(\operatorname{tr} \mathbf{V})^2 - \operatorname{tr}(\mathbf{V}^2)]/2$, $i_3 = \det \mathbf{V}$, and the constitutive equation giving the Cauchy stress tensor $\mathbf{T}$ is (**3**)

$$\mathbf{T} = p\mathbf{V} + \omega_0 \mathbf{1} + \omega_2 \mathbf{V}^2, \tag{2.2}$$

where p is an arbitrary scalar, to be found from the equations of motion and the boundary conditions; and the material response functions ω_0 and ω_2 are defined by

$$\omega_0 = \partial\Sigma/\partial i_3, \quad \omega_2 = -i_3^{-1}\partial\Sigma/\partial i_2, \tag{2.3}$$

and should satisfy the Beatty–Hayes A-inequalities (**3**)

$$\omega_0(i_2, i_3) \leqslant 0, \quad \omega_2(i_2, i_3) > 0. \tag{2.4}$$

In the case where the material is maintained in a state of finite pure homogeneous static deformation, with principal stretch ratios $\lambda_1, \lambda_2, \lambda_3$ along the x_1, x_2, x_3 axes, the Cauchy stress tensor is the constant tensor $\mathbf{T}_o$ given by

$$\mathbf{T}_o = (p_o\lambda_1 + \omega_0 + \lambda_1^2\omega_2)\mathbf{i}\otimes\mathbf{i} + (p_o\lambda_2 + \omega_0 + \lambda_2^2\omega_2)\mathbf{j}\otimes\mathbf{j} + (p_o\lambda_3 + \omega_0 + \lambda_3^2\omega_2)\mathbf{k}\otimes\mathbf{k}.$$

Here ω_0 and ω_2 are evaluated at i_2, i_3 given by

$$i_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1, \quad i_3 = \lambda_1\lambda_2\lambda_3. \tag{2.5}$$

Of course,

$$\lambda_1 + \lambda_2 + \lambda_3 = 3 \tag{2.6}$$

in order to satisfy (2.1). It is assumed that the boundary $x_2 = 0$ is free of tractions so that $T_{o22} = 0$; and that the compressive loads P_1 and P_3 are applied at $x_1 = \infty$ and $x_3 = \infty$ to maintain the deformation, so that $P_1 = -T_{o11}$ and $P_3 = -T_{o33}$. Hence,

$$p_o = -(\omega_0 + \lambda_2^2\omega_2)/\lambda_2, \quad P_\Gamma = (\lambda_2 - \lambda_\Gamma)(-\omega_0 + \lambda_\Gamma\lambda_2\omega_2)/\lambda_2 \quad (\Gamma = 1, 3). \tag{2.7}$$

2.2 *Incremental equations for surface waves*

Beatty and Hayes (**8**) wrote the general equations for small-amplitude motions in a Bell-constrained material maintained in a static state of finite pure homogeneous deformation (as described in the previous subsection). These equations were then specialized by this author (**4**) to surface (Rayleigh) waves. The infinitesimal superposed wave is of the form $\Re\{\mathbf{U}(kx_2)e^{ik(x_1-vt)}\}$, where $\mathbf{U}$ is an unknown decaying function. Hence the wave propagates in the direction of the x_1-axis with speed v and wave number k and is attenuated in the direction of the x_2-axis. The incremental tractions acting upon the planes $x_2 =$ constant are σ_{21}^* and σ_{22}^*, and the introduction of the scalar functions $t_1(kx_2)$ and $t_2(kx_2)$, defined by

$$\sigma_{21}^*(x_1, x_2, t) = kt_1(kx_2)e^{ik(x_1-vt)}, \quad \sigma_{22}^*(x_1, x_2, t) = kt_2(kx_2)e^{ik(x_1-vt)}, \tag{2.8}$$

allows for a compact and simple form of the equations of motion and of the boundary conditions. Explicitly, the equations of motion are (**4**)

$$t_1' + i\lambda_1\lambda_2^{-1}t_2 - (\lambda_1\lambda_2^{-1}C - \rho v^2)U_1 = 0, \tag{2.9}$$

$$t_2' + it_1 - [b_3(\lambda_1^2 - \lambda_2^2) - \rho v^2]U_2 = 0, \tag{2.10}$$

$$U_2' + i\lambda_1\lambda_2^{-1}U_1 = 0, \tag{2.11}$$

$$b_3\lambda_2^2U_1' + ib_3\lambda_2^2U_2 - t_1 = 0. \tag{2.12}$$

Here the prime denotes differentiation with respect to kx_2, ρ is the mass density of the material, and

$$\begin{aligned} b_3 &= \frac{-\omega_0 + \lambda_1\lambda_2\omega_2}{\lambda_2(\lambda_1+\lambda_2)} > 0, \\ C_{\alpha\beta} &= 2\lambda_\alpha^2\delta_{\alpha\beta}\omega_2 - \lambda_\beta^2(\omega_{02} + \lambda_\alpha^2\omega_{22}) + \lambda_1\lambda_2\lambda_3(\omega_{03} + \lambda_\alpha^2\omega_{23}), \\ C &= \lambda_1^{-1}\lambda_2 C_{11} + \lambda_1\lambda_2^{-1}C_{22} - C_{12} - C_{21} - 2\omega_0 - (\lambda_1^2+\lambda_2^2)\omega_2, \end{aligned} \tag{2.13}$$

where the derivatives $\omega_{0\Gamma}$, $\omega_{2\Gamma}$ ($\Gamma = 2, 3$) of the material response functions ω_0, ω_2 are taken with respect to i_Γ and evaluated at i_2, i_3 given by (2.5). Note that the quantity b_3 defined above is positive according to the A-inequalities (2.4). Finally, the boundary conditions are simply

$$t_1(0) = t_2(0) = 0. \tag{2.14}$$

2.3 *Exact secular equation and exact bifurcation criterion*

The incremental Bell constraint (2.11) suggests the introduction of a function φ defined by

$$U_1(kx_2) = i\varphi'(k\lambda_1\lambda_2^{-1}x_2), \quad U_2(kx_2) = \varphi(k\lambda_1\lambda_2^{-1}x_2). \tag{2.15}$$

With this choice, and by (2.9) and (2.12), the traction components t_1 and t_2 are expressed in terms of φ as

$$t_1 = ib_3\lambda_2^2(\lambda_1\lambda_2^{-1}\varphi'' + \varphi), \quad t_2 = -b_3\lambda_1\lambda_2\varphi''' + \lambda_1^{-1}\lambda_2(\lambda_1\lambda_2^{-1}C - b_3\lambda_1\lambda_2 - \rho v^2)\varphi', \tag{2.16}$$

where the prime now denotes derivative with respect to the argument of φ, and (2.10) reads

$$b_3\lambda_1^2\varphi'''' - (\lambda_1\lambda_2^{-1}C - 2b_3\lambda_1\lambda_2 - \rho v^2)\varphi'' + (b_3\lambda_1^2 - \rho v^2)\varphi = 0. \tag{2.17}$$

Now a law of exponential decay is chosen for φ,

$$\varphi(z) = Ae^{-s_1 z} + Be^{-s_2 z}, \quad \Re(s_i) > 0, \tag{2.18}$$

for some constants A and B. (It is implicit in the form of (2.18) that s_1 and s_2 are distinct.) By substitution into (2.17), we see that the s_i are roots of the following biquadratic:

$$(b_3\lambda_1^2)s^4 - (\lambda_1\lambda_2^{-1}C - 2b_3\lambda_1\lambda_2 - \rho v^2)s^2 + (b_3\lambda_1^2 - \rho v^2) = 0, \tag{2.19}$$

so that

$$s_1^2 + s_2^2 = (\lambda_1\lambda_2^{-1}C - 2b_3\lambda_1\lambda_2 - \rho v^2)/(b_3\lambda_1^2) \quad \text{and} \quad s_1^2 s_2^2 = (b_3\lambda_1^2 - \rho v^2)/(b_3\lambda_1^2). \tag{2.20}$$

The roots s_1^2 and s_2^2 of this real quadratic may be both real (and then they are positive because $\Re(s_i) > 0$) or both complex (and then they are conjugate because (2.19) is a real polynomial); in both cases, $s_1^2 s_2^2 \geqslant 0$, and so by $(2.20)_2$,

$$0 \leqslant v \leqslant \sqrt{b_3\lambda_1^2/\rho}. \tag{2.21}$$

The upper limit of this interval corresponds to the speed of a bulk shear wave propagating along the x_1-direction.

Now the boundary conditions (2.14), used in conjunction with (2.16) and $(2.20)_1$, are

$$(\lambda_1\lambda_2^{-1}s_1^2+1)A+(\lambda_1\lambda_2^{-1}s_2^2+1)B=0, \quad s_1(\lambda_1\lambda_2^{-1}s_2^2+1)A+s_2(\lambda_1\lambda_2^{-1}s_1^2+1)B=0,$$

and the vanishing of the determinant for this linear homogeneous system of two equations gives the *exact secular equation*

$$[b_3(\lambda_1^2-\lambda_2^2)-\rho v^2]\sqrt{b_3\lambda_1^2}+(\lambda_1\lambda_2^{-1}C-\rho v^2)\sqrt{b_3\lambda_1^2-\rho v^2}=0. \tag{2.22}$$

In the process, we used (2.20) and dropped the factor s_1-s_2. Note that by bringing the second term of (2.22) to the right-hand side and squaring, we obtain the cubic secular equation (**4**), which has spurious roots. Now for certain stretch ratios λ_1, λ_2, λ_3, the speed v tends to zero in (2.22) and the exact *bifurcation criterion* is deduced as

$$b_3(\lambda_1^2-\lambda_2^2)+\lambda_1\lambda_2^{-1}C=0. \tag{2.23}$$

This equation defines a surface in the space of the stretch ratios which separates a region where the homogeneous deformations of the Bell half-space are always stable from a region where they might be unstable. Of course, the critical stretch ratios must also satisfy the Bell constraint (2.6).

We now recast the secular equation for surface waves in a polynomial form for the positive quantity η, defined by $\eta=\{1-(\rho v^2)/(b_3\lambda_1^2)\}^{\frac{1}{2}}$ (**9**), as

$$f(\eta)\equiv\eta^3+\eta^2+\{C/(b_3\lambda_1\lambda_2)-1\}\eta-\lambda_1^{-2}\lambda_2^2=0. \tag{2.24}$$

Clearly, at $\eta=0$ (corresponding to a transverse bulk wave), we have $f(0)=-\lambda_1^{-2}\lambda_2^2<0$; at $\eta=1$ (corresponding to $v=0$), the secular equation tends to the bifurcation criterion $f(1)=[b_3(\lambda_1^2-\lambda_2^2)+\lambda_1\lambda_2^{-1}C]/(b_3\lambda_1^2)=0$.

Up to this point, the setting has been that of incremental surface motions and deformations for a general Bell-constrained half-space, maintained in a static state of arbitrary pure homogeneous triaxial stretch. More results may actually be obtained in this general setting regarding the conditions of existence and the uniqueness of a Rayleigh wave; this is done elsewhere (**10**). We now turn our attention to two specific types of Bell materials and compare the results obtained in plane and in equibiaxial prestrains with those obtained for rubber-like incompressible materials.

3. Specific forms of strain-energy densities

3.1 *Simple hyperelastic Bell materials*

For *simple hyperelastic Bell materials* (**3**), the strain-energy function Σ_{SHB} is given by

$$\Sigma_{\mathrm{SHB}}=\mathcal{C}_1(3-i_2)+\mathcal{C}_2(1-i_3), \tag{3.1}$$

where $\mathcal{C}_1$ and $\mathcal{C}_2$ are positive constants. The material response functions ω_0 and ω_2 and the quantities b_3 and C provided by (2.3) and (2.13) are now

$$\omega_0=-\mathcal{C}_2, \quad \omega_2=\mathcal{C}_1/i_3, \quad b_3=\frac{\mathcal{C}_2+\mathcal{C}_1\lambda_3^{-1}}{\lambda_2(\lambda_1+\lambda_2)}, \quad C=2(\mathcal{C}_2+\mathcal{C}_1\lambda_3^{-1}). \tag{3.2}$$

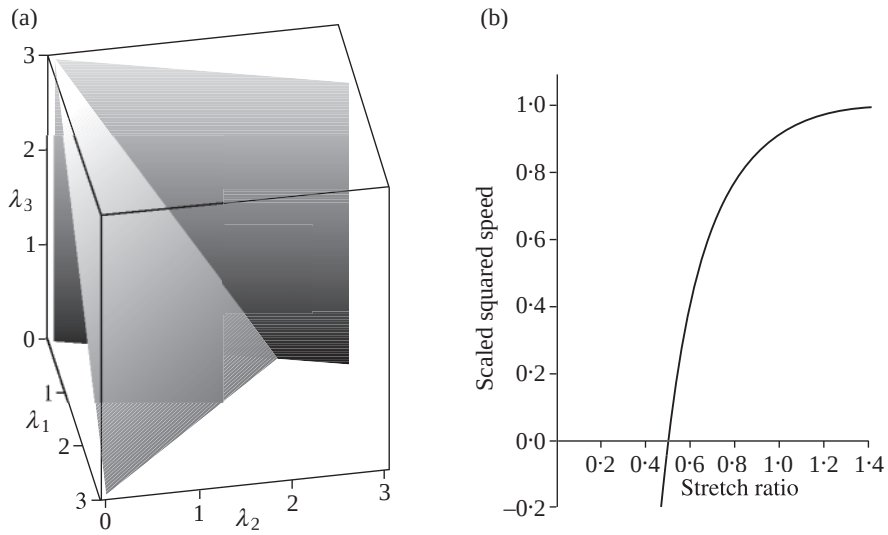

Fig. 1 Near-the-surface stability for 'simple hyperelastic Bell' materials. (a) Region of stability; (b) surface wave speed

In that context, the bifurcation criterion (2.23) simplifies considerably to

$$3\lambda_1 - \lambda_2 = 0, \tag{3.3}$$

which is a particularly simple linear relationship between the stretch ratios λ_1 and λ_2. This bifurcation criterion is universal to the whole class of simple hyperelastic Bell materials because it does not depend on $\mathcal{C}_1, \mathcal{C}_2$. This equation delimits a plane in the stretch ratios space $(\lambda_1, \lambda_2, \lambda_3)$, which cuts the constraint plane (2.6) along the straight segment going from the point (0, 0, 3) to the point $(\frac{3}{4}, \frac{9}{4}, 0)$. Moreover, the analysis below shows that the region which is stable with respect to incremental perturbations (where there exists a root $\rho v^2 > 0$ to the secular equation) is $3\lambda_1 - \lambda_2 > 0$. In Fig. 1(a), the plane (3.3) cuts the triangle of the possible values for the stretch ratios (2.6) into two parts, of which the visible one is the region of linear surface stability of any simple hyperelastic Bell material.

For the propagating surface wave, we write the secular equation (2.24) in terms of η as

$$f(\eta) = \eta^3 + \eta^2 + (1 + 2\lambda_1^{-1}\lambda_2)\eta - \lambda_1^{-2}\lambda_2^2 = 0. \tag{3.4}$$

As noted in the general case, $f(0) < 0$. At the other end of the interval (2.21), $f(1) = (3\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2)\lambda_1^{-2}$. So, because f is a monotone increasing function for $\eta > 0$, there exists a root to the secular equation (3.4) in the interval [0, 1] if and only if $3\lambda_1 - \lambda_2 > 0$; moreover, the root is unique.

In Fig. 1(b), the influence of the prestrain upon the speed of the surface wave is illustrated in the case of plane strain ($\lambda_3 = 1$). On the abscissa, λ_1 is increased from a compressive value ($\lambda_1 < 1$) to a tensile value ($\lambda_1 > 1$). The coordinate on the ordinate is the squared surface wave speed, scaled

with respect to the transverse bulk wave speed, that is, $\rho v^2/(\mu\lambda_1^2)$. At $\lambda_1 = 1$, the half-space is isotropic ($\lambda_1 = \lambda_2 = \lambda_3 = 1$) because of (2.6), and the scaled squared speed is equal to 0·9126, the value found by Lord Rayleigh (**11**) in the incompressible linear isotropic case. Under an increasing compressive load ($P_1 > 0$, $\lambda_1 < 1$), the surface wave speed decreases until the critical stretch of $(\lambda_1)_{\mathrm{cr}} = 0{\cdot}5$ (see section 3.3). Conversely, under a tensile load ($P_1 < 0$, $\lambda_1 > 1$), the surface wave speed increases, with the speed of the transverse bulk wave as an upper bound.

3.2 *Bell's empirical model*

For *Bell's empirical model materials* (**6**), the strain-energy function Σ_{BEM} is given by

$$\Sigma_{\mathrm{BEM}} = \tfrac{2}{3}\beta_0[2(3 - i_2)]^{\frac{3}{4}}, \tag{3.5}$$

where β_0 is a positive constant. The material response functions ω_0 and ω_2 and the quantities b_3 and C provided by (2.3) and (2.13) are now

$$\omega_0 = 0, \quad \omega_2 = i_3^{-1}\beta_0[2(3 - i_2)]^{-\frac{1}{4}}, \quad b_3 = \frac{\lambda_1\omega_2}{\lambda_1 + \lambda_2}, \quad C = \left[2 - \frac{(\lambda_1 - \lambda_2)^2}{4(3 - i_2)}\right]\lambda_1\lambda_2\omega_2.$$

In that context, the bifurcation criterion (2.23) may be arranged as

$$3 - \frac{(\lambda_1 - \lambda_2)^2}{4(3 - i_2)} - \lambda_1^{-1}\lambda_2 = 0, \tag{3.6}$$

which is the stability equation (7.10) of Beatty and Pan (**12**) for this problem; it is actually a cubic for λ_1 and λ_2. It delimits a surface in the stretch ratios space $(\lambda_1, \lambda_2, \lambda_3)$, which cuts the constraint plane (2.6) into a part which is 'unstable' (in the linearized theory) and a part which is always stable (in the linearized theory). This partition of the triangle of possible stretch ratios is visible on Fig. 2(a). By extension from the plane strain case where $\lambda_3 = 1$ (treated below), we deduce that the visible part of the triangle is the stable one. For a clearer picture, Fig. 2(b) (where the plane of the figure coincides with the plane of the triangle) shows the intersection between the triangle and the bifurcation curve.

For the propagating surface wave we consider only the case where the underlying deformation is a plane strain such that $\lambda_3 = 1$. Then the Bell constraint (2.6) reduces to $\lambda_1 + \lambda_2 = 2$, and neither λ_1 nor λ_2 may be greater than 2. Then the secular equation (2.24) reduces to

$$f(\eta) = \eta^3 + \eta^2 + \lambda_1^{-1}(2 - \lambda_1)\eta - \lambda_1^{-2}(2 - \lambda_1)^2 = 0. \tag{3.7}$$

As noted in the general case, $f(0) < 0$. On the other hand, $f(1) = 2(3\lambda_1 - 2)/\lambda_1^2$, and so, because f is a monotone increasing function for $\eta > 0$, there exists a root to the secular equation (3.7) in the interval [0, 1] if and only if $3\lambda_1 - 2 > 0$; moreover, the root is unique.

3.3 *Comparisons with incompressible rubber*

The stability of a deformed half-space made of incompressible rubber was first studied by Biot. He used the neo-Hookean model but noted (**2**, p. 165) that the results were also valid for the Mooney–Rivlin model (Flavin (**13**) solved explicitly this latter case). Biot obtained the bifurcation criterion,

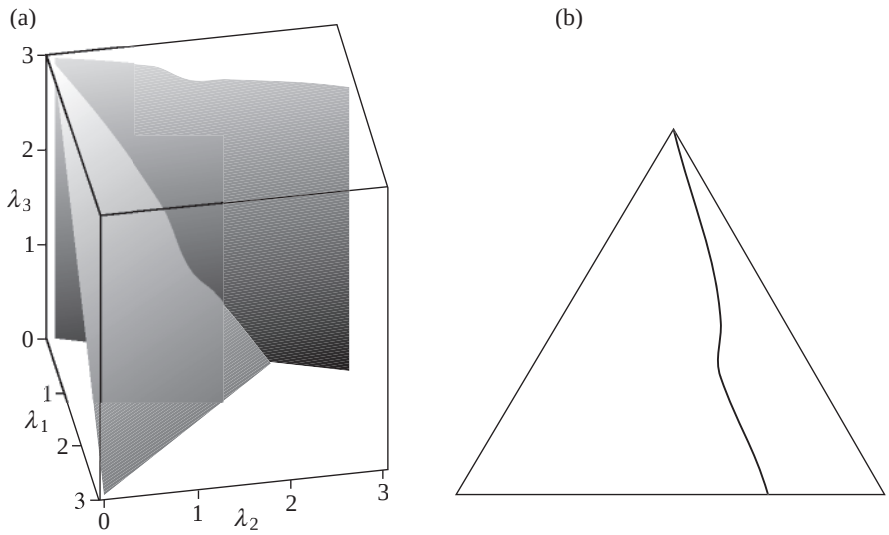

Fig. 2 Near-the-surface stability for 'Bell empirical model' materials. (a) Region of stability; (b) bifurcation curve in triangle

showed that it is universal relative to both classes of materials, and computed the value of the critical stretch $(\lambda_1)_{\mathrm{cr}}$ at which the rubber half-space becomes 'unstable' under compressive loads, first in the case of plane strain $\lambda_3 = 1$, then in the case of the biaxial strain $\lambda_2 = \lambda_3$. Also, Green and Zerna (**14**, p. 137) found the critical stretch for the biaxial prestrain $\lambda_1 = \lambda_3$. In each case a cubic must be solved in order to evaluate the critical stretch. In the case of a general triaxial prestrain, the bifurcation criterion is

$$\lambda_1^3 + \lambda_1^2\lambda_2 + 3\lambda_1\lambda_2^2 - \lambda_2^3 = 0. \tag{3.8}$$

We now show that for Bell's empirical models and for simple hyperelastic Bell materials, the critical stretch can be found explicitly.

First we let the half-space made of Bell-constrained material be deformed in such a way that there is no extension in the x_3-direction ($\lambda_3 = 1$). According to $(2.7)_{2,3}$, this deformation is possible when the loads $P_1 = 2(1-\lambda_1)[\lambda_1\omega_2 - \omega_0/(2-\lambda_1)]$ and $P_3 = (1-\lambda_1)[\omega_2 - \omega_0/(2-\lambda_1)]$ are applied at infinity. Then $\lambda_2 = 2 - \lambda_1$ by (2.6). For Bell's empirical model and for simple hyperelastic Bell materials, the bifurcation criterion (3.3) reduces respectively to

$$3(\lambda_1)_{\mathrm{cr}} - 2 = 0 \quad \text{and} \quad 4(\lambda_1)_{\mathrm{cr}} - 2 = 0. \tag{3.9}$$

Next we let the half-space made of Bell-constrained material expand freely in the x_3-direction, so that $P_3 = 0$. Then we have $\lambda_2 = \lambda_3 = (3-\lambda_1)/2$ by $(2.7)_3$ and (2.6), and the load $P_1 = (3/2)(1-\lambda_1)[\lambda_1\omega_2 - 2\omega_0/(3-\lambda_1)]$ must be applied at infinity to maintain the deformation. For Bell's empirical model and for simple hyperelastic Bell materials, the bifurcation criterion (3.3) reduces respectively to

$$11(\lambda_1)_{\mathrm{cr}} - 6 = 0 \quad \text{and} \quad 7(\lambda_1)_{\mathrm{cr}} - 3 = 0. \tag{3.10}$$

Table 1 Critical stretch ratios $(\lambda_1)_{\mathrm{cr}}$ for surface instability

	Bell empirical	rubber	simple Bell
$\lambda_1 = \lambda_3$	0·706	0·666	0·600
$\lambda_3 = 1$	0·667	0·544	0·500
$\lambda_2 = \lambda_3$	0·545	0·444	0·429

Finally we consider that the Bell material is subject to a biaxial prestrain such that $\lambda_1 = \lambda_3$. Then $\lambda_2 = 3 - 2\lambda_1$ and $P_1 = P_3 = 3(1-\lambda_1)[\lambda_1\omega_2 - \omega_0/(3-2\lambda_1)]$. For Bell's empirical model and for simple hyperelastic Bell materials, the bifurcation criterion (3.3) reduces respectively to

$$17(\lambda_1)_{\mathrm{cr}} - 12 = 0 \quad \text{and} \quad 5(\lambda_1)_{\mathrm{cr}} - 3 = 0. \tag{3.11}$$

In Table 1, the numerical values for the critical stretches are given for the classes of Bell's empirical model (second column), of neo-Hookean and Mooney–Rivlin incompressible materials (**2**, **14**) (third column), and of simple hyperelastic Bell materials (fourth column), in the cases of plane strain (third row) and of biaxial strain (second and fourth rows). It appears that rubber can be compressed more than Bell's empirical model but less than simple hyperelastic Bell materials before it loses its near-the-surface stability (in the linearized theory).

3.4 *A note on relative-universal bifurcation criteria*

The strain-energy functions for the Mooney–Rivlin model and for the simple hyperelastic Bell material depend both upon two distinct material constants:

$$\Sigma_{\mathrm{MR}} = \mathcal{D}_1(I_1 - 3) + \mathcal{D}_2(I_2 - 3), \quad \Sigma_{\mathrm{SHB}} = \mathcal{C}_1(3 - i_2) + \mathcal{C}_2(1 - i_3), \tag{3.12}$$

respectively, where $\mathcal{D}_1$, $\mathcal{D}_2$ are constants and I_1, I_2 are the first two invariants of the left Cauchy–Green tensor $\mathbf{B} = \mathbf{V}^2$. The fact that their bifurcation criteria are 'relative-universal' (**15**) to each class might come as a surprising result, but is easily understood once the strain-energy functions are written in terms of the principal stretches of the deformation (**9**) as $\Sigma_{\mathrm{MR}}(I_1, I_2) \equiv W_{\mathrm{MR}}(\lambda_1, \lambda_2, \lambda_3)$ and $\Sigma_{\mathrm{SHB}}(i_2, i_3) \equiv W_{\mathrm{SHB}}(\lambda_1, \lambda_2, \lambda_3)$, where

$$W_{\mathrm{MR}} = \mathcal{D}_1(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3) + \mathcal{D}_2(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2 - 3), \tag{3.13}$$

$$W_{\mathrm{SHB}} = \mathcal{C}_1(3 - \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1) + \mathcal{C}_2(1 - \lambda_1\lambda_2\lambda_3). \tag{3.14}$$

Indeed, the bifurcation criterion for a general incompressible hyperelastic half-space may be written in terms of the first and second derivatives of its strain-energy function W with respect to λ_i $(i = 1, 2)$ as (**9**)

$$\lambda_2[W_1 + (2 - \lambda_1^{-1}\lambda_2)W_2] + \lambda_1^2 W_{11} - 2\lambda_1\lambda_2 W_{12} + \lambda_2^2 W_{22} = 0. \tag{3.15}$$

When W is specialized to the Mooney–Rivlin form (3.13), it yields the relative-universal bifurcation

criterion (3.8). In fact, many subclasses of incompressible materials have relative-universal bifurcation criteria. For instance, any incompressible material with the following strain-energy function:

$$W = \mathcal{D}_1(\lambda_1^n + \lambda_2^n + \lambda_3^n - 3) + \mathcal{D}_2(\lambda_1^n\lambda_2^n + \lambda_2^n\lambda_3^n + \lambda_3^n\lambda_1^n - 3) \tag{3.16}$$

(where $n = 1, 2, 3, \ldots$), has the following relative-universal bifurcation criterion:

$$(n-1)\lambda_1^{n+1} + \lambda_1^n\lambda_2 + (n+1)\lambda_1\lambda_2^n - \lambda_2^{n+1} = 0. \tag{3.17}$$

In particular, the bifurcation criterion (3.3), which happens to coincide with (3.17) when $n = 1$, is also valid for incompressible Varga materials ($n = 1$ in (3.16)).

Turning back to Bell-constrained materials, we note that it is a simple matter to write the quantities b_3 and C in (2.13) in terms of the derivatives of $W(\lambda_1, \lambda_2, \lambda_3)$. We find that

$$b_3 = \frac{W_1 - W_2}{\lambda_2\lambda_3(\lambda_1^2 - \lambda_2^2)}, \quad C = \frac{W_{11} - 2W_{12} + W_{22}}{\lambda_3}, \tag{3.18}$$

so that the bifurcation criterion (2.23) for Bell materials is rewritten as

$$W_1 - W_2 + \lambda_1(W_{11} - 2W_{12} + W_{22}) = 0. \tag{3.19}$$

When W is specialized to the simple hyperelastic Bell model (3.14), it yields the relative-universal bifurcation criterion (3.3). Similarly, when W is specialized to the strain-energy function of the Bell empirical model (3.5), written as

$$W_{\mathrm{BEM}} = \tfrac{2}{3}\beta_0[2(3 - \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)]^{\frac{3}{4}}, \tag{3.20}$$

it yields the bifurcation criterion (3.6).

4. Concluding remarks on stability and on the Bell constraint

A word of caution is needed, in conclusion, regarding the notion of instability. Throughout the paper, care has been taken to talk of instability 'in the linearized theory' and of a region in the stretch ratios space where the deformed half-space 'might be unstable'. This is so because conclusions about the actual stability of a finitely deformed half-space do not necessarily come out of the dynamical method of surface wave analysis. As Chadwick and Jarvis (**16**) pointed out, 'the exponential growth of a solution obtained . . . on the basis of a linearized theory eventually violates the assumption underlying the linearization'. Some authors have linked instability analysis and bifurcation theory, but as Guz remarked (**17**, p. 268), such a comparative analysis 'is only qualitative and to a certain extent sketchy'. Finally, Biot (**2**) adopted a static approach to the problem and found that at the critical stretch, the surface deflection (that is, the component of the displacement normal to the surface) became infinite; however, it is clear that such a deformation can hardly be called 'incremental'. Nevertheless, it is comforting to remark that each approach yields the same result for the critical stretches, and to know that, to some extent, concording experimental results exist (**18**). On the other hand, the linear theory of stability has its limits. As kindly pointed out by a referee, Chadwick and Jarvis (**16**) go on to say that in a situation where exponential growth occurs, 'further enquiry is needed to discover whether or not the terms initially neglected cause the motion to be

stabilized'. Such a line of inquiry has indeed been followed ever since (see for instance Fu (**19**) or Fu and Rogerson (**20**)), with the result that in certain cases, terms neglected in the linear theory do indeed contribute to a greater stability.

Finally, the motivation for the use of the Bell constraint is now exposed. Using his countless experiments, James F. Bell produced a large literature corroborating the existence for certain metals of the constraint that now bears his name. A thorough background and detailed account of his research is given in a recent review by Beatty (**21**). Note, however, that the actual existence of 'Bell materials' is controversial and that Bell's results have been criticized (**22**, **23**). From a theoretical point of view there is justification in studying classical problems (such as the one presented here) for a material subject to an internal constraint other than incompressibility. Indeed incompressibility is an exceptional isotropic constraint because the corresponding reaction stress is spherical. This property distinguishes incompressibility among generic isotropic constraints. For instance Pucci and Saccomandi (**24**) proved that the deformation (**25**)

$$x_1 = AX_1 + \sin\lambda X_2, \quad x_2 = DX_2, \quad x_3 = AX_3 - \cos\lambda X_2,$$

where A, D, λ are constants, is universal for all isotropically constrained materials, *apart* from incompressible materials. Also, it is always possible to subject a constrained material successively to a triaxial stretch followed by a simple shear:

$$x_1 = \lambda_1 X_1 + k\lambda_2 X_2, \quad x_2 = \lambda_2 X_2, \quad x_3 = \lambda_3 X_3,$$

where the λ_i and k are constants, in such a way that the following relations for the Cauchy stress components are universal: $\sigma_{11} = \sigma_{22}$ and $\sigma_{12} = 0$, *except* when the reaction stress is spherical (these and related points are developed in detail by Saccomandi in (**15**, **26**)). Therefore, a better understanding of the (theoretical) behaviour of Bell constrained materials gives us a better understanding of isotropic constraints and of their general—and not exceptional—mechanical properties.

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