Rayleigh Waves in Anisotropic Crystals Rotating About the Normal to a Symmetry Plane

M. Destrade
Laboratoire de Modélisation en Mécanique,
UMR 7607, CNRS,
Université Pierre et Marie Curie,
4 place Jussieu, Tour 66, case 162,
75252 Paris Cedex 05, France
e-mail: destrade@lmm.jussieu.fr

The propagation of surface acoustic waves in a rotating anisotropic crystal is studied. The crystal is monoclinic and cut along a plane containing the normal to the symmetry plane; this normal is also the axis of rotation. The secular equation is obtained explicitly using the "method of the polarization vector," and it shows that the wave is dispersive and decelerates with increasing rotation rate. The case of orthorhombic symmetry is also treated. The surface wave speed is computed for 12 monoclinic and 8 rhombic crystals, and for a large range of the rotation rate/wave frequency ratio.
[DOI: 10.1115/1.1756140]

1 Introduction

Introduced more than 30 years ago, surface acoustic wave (SAW) devices have been used with great success by the telecommunication industry: nowadays, they are produced in large quantities (several billions per year) and used in wireless transmission and reception technology for color television sets, cell phones, global positioning systems, etc. In recent years, new applications for SAW devices have emerged, namely acoustic sensors, which are passive (no power supply is needed), resistant, almost nonaging, cheap (only one photo-lithographic process is involved in the production), light (less than 1g) and can be operated remotely and wirelessly. For instance [1], SAW identification tags are used for highway toll collection in Norway and for the Munich subway system; SAW temperature sensors can achieve a resolution of 0.02°C from −196°C up to 500°C; wide ranges and fine resolutions are also achieved for pressure, torque, or current sensors, etc. Also, the automotive industry is engaged in the search for an "intelligent tire" which could provide direct information on its current state as the car is moving; in this context SAW sensors have been used to measure tire pressure, [2], or friction, [3], as the wheel rotates. In general, SAW devices may be used as angular rate sensors (gyroscopes) to measure frequency shifts due to the rotation, [4–6]. In the present paper, an investigation of the effect of rotation upon the speed of surface (Rayleigh) waves in an anisotropic crystal is presented.

The crystal may possess as little as a single plane of symmetry. It is cut along any plane containing the normal to the symmetry plane and is assumed to rotate at a constant rate about this normal. The surface wave is polarized in the symmetry plane. In other words, it suffices to consider the propagation of a surface wave in the x_1-direction of a monoclinic crystal with symmetry plane at $x_3=0$, cut along the $x_2=0$ plane, and rotating about the x_3-axis (see Fig. 1). The secular equation for rotating materials was obtained by others but in simpler settings: by Clarke and Burdess in an isotropic material, first for small rotation rate/wave frequency ratios, [4], then for any ratio, [5]; by Grigor'evskii, Gulyaev, and Kozlov [7] also for isotropic materials but neglecting the centrifugal force; and by Fang, Yang, and Jiang [6] for crystals having tetragonal symmetry. Here, the analysis is fully developed for crystals with a single symmetry plane, up to the derivation of the secular equation in explicit form, that is an equation giving the Rayleigh wave speed in terms of the elastic parameters and of the rotation rate.

The equation is reached in Sec. 3, after the governing equations have been written down in Sec. 2. The secular equation turns out to be a polynomial of degree 8 for the squared wave speed and also for the squared rotation rate/wave frequency ratio. In the simpler case of orthorhombic symmetry (Sec. 4), the polynomial is of degree 6. The Rayleigh wave speed is computed numerically for 20 specific anisotropic materials (12 monoclinic, 8 orthorhombic) and for a rotation rate/wave frequency ratio varying from 0 to 10. Of course, this range is way beyond the elastic behavior limit, and is irrealistic for pratical purposes where the frequency of a SAW device is typically in the 100 kHz–10 MHz range. It is presented to show that the method of resolution is exact and not approximate, applies for any rate of rotation, and that in contrast with the nonrotating case, the secular equation is dispersive. At small rotation rates, and for certain crystals such as PZT-5, other papers, [6,8], show that the Rayleigh wave speed may at first increase slightly with the rotation frequency/wave frequency ratio. At large ratios, it is seen here that the wave speed decreases with increasing ratios. These variations are crucial to the understanding and correct design of rotating SAW sensors or SAW signal processing devices. A recent article, [9], describes the manufacturing of a 1 cm×1 cm SAW gyroscope and how the rotation rate may be measured using SAW technology. Another example that springs to mind is that of "spinning missiles," [10], for which one might speculate that the communication is ensured via SAW generation and processing of high-frequency signals modified by the rotation. Finally in Sec. 5, the merits of several methods of derivation for the secular equation in non-rotating crystals are discussed. This paper aims to provide a theoretical and analytical framework for the study of surface acoustic waves in rotating crystals.

2 Basic Equations

We consider a half-space $x_2 \geqslant 0$ occupied by a homogeneous anisotropic crystal possessing one plane of symmetry at $x_3=0$, and rotating at a constant angular velocity Ω about the x_3-axis. We study the propagation of a surface (Rayleigh) wave in the x_1-direction, with attenuation in the x_2-direction. In the rotating Cartesian frame $(Ox_1, Ox_2, Ox_3) \equiv (O, \mathbf{i}, \mathbf{j}, \mathbf{k})$, the equations of motion are, [11],

$$\operatorname{div} \boldsymbol{\sigma} = \rho \mathbf{u}_{,tt} + 2\rho\Omega \mathbf{k}\times\mathbf{u}_{,t} + \rho\Omega^2 \mathbf{k}\times(\mathbf{k}\times\mathbf{u}), \tag{1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, ρ is the constant mass density of the material, and the comma denotes differentiation. The sec-

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the Applied Mechanics Division, January 22, 2003; final revision, September 29, 2003. Associate Editor: R. C. Benson. Discussion on the paper should be addressed to the Editor, Prof. Robert M. McMeeking, Journal of Applied Mechanics, Department of Mechanical and Environmental Engineering, University of California–Santa Barbara, Santa Barbara, CA 93106-5070, and will be accepted until four months after final publication in the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

 Copyright © 2004 by ASME

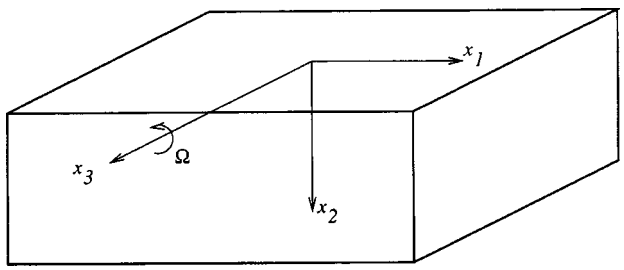

Fig. 1 Monoclinic crystal with symmetry plane at $x_3=0$, cut along $x_2=0$, and rotating about x_3 at constant angular velocity $\boldsymbol{\Omega}$

ond term in the right-hand side of Eq. (1) is due to the Coriolis acceleration, the third is due to the centrifugal acceleration. Note that Eq. (1) represents the time-dependent part of the full equations of motion. The time-independent part, namely $\operatorname{div}\boldsymbol{\sigma}^s = \rho\Omega^2\mathbf{k}\times[\mathbf{k}\times(\mathbf{u}^s+\mathbf{x})]$, where $\mathbf{u}^s=\mathbf{u}^s(\mathbf{x})$ and $\sigma^s_{ij}=c_{ijkl}u^s_{l,k}$ must be solved separately. The questions remain of (a) whether an actual time-independent solution exists for all Ω and if it does, of (b) whether the boundary conditions of a traction-free rotating half-space may be satisfied without perturbating the time-dependent boundary value problem. These questions do not seem to have been addressed in the literature, but some preliminary work seem however to suggest that (a) and (b) may be answered positively, at least within the framework of small amplitude waves superimposed upon a large elastic deformation.

Now, turning back to the time-dependent Eq. (1), the mechanical displacement $\mathbf{u}$ is taken in the form

$$\mathbf{u}(x_1,x_2,x_3,t)=\mathbf{U}(kx_2)e^{ik(x_1-vt)}, \tag{2}$$

showing a sinusoidal propagation with speed v and wave number k in the x_1-direction, and the possibility of an attenuation in the x_2-direction through the unknown function $\mathbf{U}(kx_2)$.

We wish to describe the influence of the frame rotation upon the speed of Rayleigh waves, and to this end, we introduce the following quantities:

$$X=\rho v^2, \quad \delta=\Omega/(kv)=\Omega/\omega,$$

where ω is the real frequency of the wave.

For two-dimensional motions ($\partial\mathbf{u}/\partial x_3=0$) such as Eq. (2), the anisotropy of a crystal possessing $x_3=0$ as a symmetry plane is described by the following strain-stress relationship, [12]:

$$\begin{bmatrix}\epsilon_{11}\\ \epsilon_{22}\\ 2\epsilon_{23}\\ 2\epsilon_{31}\\ 2\epsilon_{12}\end{bmatrix}=\begin{bmatrix} s'_{11} & s'_{12} & 0 & 0 & s'_{16}\\ & s'_{22} & 0 & 0 & s'_{26}\\ & & s'_{44} & s'_{45} & 0\\ & & & s'_{55} & 0\\ & & & & s'_{66}\end{bmatrix}\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \sigma_{23}\\ \sigma_{31}\\ \sigma_{12}\end{bmatrix},$$

where the strain components ϵ_{ij} are defined in terms of the displacement components by: $2\epsilon_{ij}=u_{i,j}+u_{j,i}$, and the s'_{ij} are the reduced compliances. Alternatively, the equivalent strain-stress relations can be used, ([12], p. 39),

$$\boldsymbol{\sigma}^\circ=\mathbf{C}^\circ\boldsymbol{\epsilon}^\circ, \quad \mathbf{C}^\circ\mathbf{s}'=\mathbf{1}, \tag{3}$$

where $\boldsymbol{\sigma}^\circ=[\sigma_{11},\sigma_{22},\sigma_{23},\sigma_{31},\sigma_{12}]^{\mathrm{T}}$, $\boldsymbol{\epsilon}^\circ=[\epsilon_{11},\ldots,2\epsilon_{12}]^{\mathrm{T}}$. The C°_{ij} are elements, in the Voigt notation, of the fourth-order elastic stiffness tensor C_{ijkl}. Table 1 shows the relevant reduced compliances of 12 different monoclinic crystals, computed from the corresponding stiffnesses as collected by Chadwick and Wilson [13]; the last column gives the corresponding Rayleigh wave speed in the nonrotating case, [14].

In view of the form Eq. (2) for the displacement, we introduce the functions t_1, t_2 for the tractions σ_{12}, σ_{22} on the planes x_2 = const. as

$$\sigma_{12}(x_1,x_2,x_3,t)=ikt_1(kx_2)e^{ik(x_1-vt)},$$

$$\sigma_{22}(x_1,x_2,x_3,t)=ikt_2(kx_2)e^{ik(x_1-vt)}.$$

Then, substituting Eqs. (2) and (3) into the equations of motion (1), we derive the following system of linear first-order differential equations for U_1, U_2, t_1, t_2,

$$\begin{bmatrix}\mathbf{U}'\\ \mathbf{t}'\end{bmatrix}=i\begin{bmatrix}\mathbf{N}_1 & \mathbf{N}_2\\ \check{\mathbf{N}}_3+(1+\delta^2)X\mathbf{1} & \mathbf{N}_1\end{bmatrix}\begin{bmatrix}\mathbf{U}\\ \mathbf{t}\end{bmatrix}, \tag{4}$$

where $\mathbf{U}=[U_1,U_2]^{\mathrm{T}}$, $\mathbf{t}=[t_1,t_2]^{\mathrm{T}}$, and the prime denotes differentiation with respect to kx_2. Note that, as in the static case, [15], the antiplane strain (stress) decouples from the plane strain (stress) and need not be considered for this problem. This decoupling would not occur if the crystal was rotating about the x_1-axis or the x_2-axis, [6].

The surface $x_2=0$ is free of tractions and so, the boundary conditions are

$$t_1(0)=t_2(0)=0. \tag{5}$$

In Eq. (4), $\mathbf{N}_1$ and $\mathbf{N}_2$ are the same as the 2×2 submatrices of the 6×6 fundamental elasticity matrix $\mathbf{N}$ from Ingebrigsten and Tonning [16]. Their real matrix $\mathbf{N}_3$, however, has been modified by the introduction of off-diagonal pure imaginary terms. Explicitly, we have

Table 1 Values of the reduced compliances (10^{-12} m^2/N), density (kg/m^3), and (nonrotating) surface wave speed (m/s) for 12 monoclinic crystals

Material	s'_{11}	s'_{22}	s'_{12}	s'_{16}	s'_{26}	s'_{66}	ρ	v_R
diphenyl	854	1858	−366	−698	−1.44	5049	1114	1276
tin fluoride	345	228	−59.2	−197	120	922	4875	1339
tartaric acid	343	211	−164	−223	301	1650	1760	1756
oligoclase	133	227	−108	97.0	−160	483	2638	2413
microcline	94.5	165	−35.1	47.2	1.69	446	2561	2816
gypsum	243	130	−68.6	32.9	28.1	326	2310	3011
hornblende	63.3	103	−32.7	−15.8	−2.72	320	3120	3049
aegirite-augite	53.6	78.4	−21.0	−10.6	−33.5	237	3420	3382
epidote	53.3	49.6	−11.3	17.7	−3.74	237	3400	3409
augite	54.5	64.4	−19.5	−19.0	−15.7	211	3320	3615
diopside	53.1	58.6	−20.1	24.0	6.98	186	3310	3799
diallage	49.8	69.1	−11.3	−6.88	−14.5	166	3300	4000

$$-\mathbf{N}_1=\begin{bmatrix} r_6 & 1 \\ r_2 & 0 \end{bmatrix}, \quad \mathbf{N}_2=\begin{bmatrix} n_{66} & n_{26} \\ n_{26} & n_{22} \end{bmatrix}, \quad -\check{\mathbf{N}}_3=\begin{bmatrix} \eta & 2i\delta X \\ -2i\delta X & 0 \end{bmatrix},$$

where the quantities r_2, r_6, n_{22}, n_{26}, n_{66}, η are given in terms of the elastic parameters as ([14,17])

$$\eta=\frac{1}{s'_{11}}, \quad r_6=-\frac{s'_{16}}{s'_{11}}, \quad r_2=-\frac{s'_{12}}{s'_{11}},$$

$$n_{66}=\frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{16} \\ s'_{16} & s'_{66} \end{vmatrix}, \quad n_{22}=\frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{12} \\ s'_{12} & s'_{22} \end{vmatrix}, \quad n_{26}=\frac{1}{s'_{11}}\begin{vmatrix} s'_{11} & s'_{16} \\ s'_{12} & s'_{26} \end{vmatrix}.$$

Thus the rotation of the crystal perturbs the equations of motion in three ways: the introduction of dispersion through δ; a shift of magnitude δ^2 in $X=\rho v^2$ for the lower left submatrix of $\mathbf{N}$ proportional to the 2×2 unit matrix; and the modification of $\mathbf{N}_3$, which is diagonal in the nonrotating case (note that the new matrix $\check{\mathbf{N}}_3$ is Hermitian: $\overline{\check{\mathbf{N}}_3}=\check{\mathbf{N}}_3^{\mathrm{T}}$.) Despite these modifications, the secular equation can be obtained explicitly for the surface wave speed, using a method proposed by Currie [18] and by Taziev [19] for nonrotating anisotropic crystals with and without a plane of symmetry, respectively.

3 Secular Equation

The method of the polarization vector was first presented by Currie [18] to derive the secular equation for Rayleigh waves in the symmetry plane of monoclinic nonrotating crystals. Then Taziev [19] generalized the method to triclinic (no symmetry plane) crystals. This method takes advantage of the Cayley-Hamilton theorem for the fundamental matrix $\mathbf{N}$, which implies that only n matrices $\mathbf{N}^k$ $(k=1,\ldots,n)$ are linearly independent ($n=3$ for monoclinic crystals, $n=5$ for triclinic crystals). Currie used the matrices $\mathbf{N}$, $\mathbf{N}^2$, $\mathbf{N}^3$; Taziev, the matrices $\mathbf{N}$, $\mathbf{N}^2$, $\mathbf{N}^3$, $\mathbf{N}^4$, $\mathbf{N}^5$. Recently, Ting [20] placed their results within the context of the Stroh-Barnett-Lothe formalism and improved on them by showing that the choices of $\mathbf{N}^{-1}$, $\mathbf{N}$, $\mathbf{N}^2$ for monoclinic crystals and of $\mathbf{N}^{-2}$, $\mathbf{N}^{-1}$, $\mathbf{N}$, $\mathbf{N}^2$, $\mathbf{N}^3$ for triclinic crystals lead to simpler and more explicit secular equations. His approach is now adapted to our present context of a rotating crystal with one symmetry plane. An alternative derivation, not based on the Stroh-Barnett-Lothe formalism, is available elsewhere, [21,22].

We seek solutions to the equations of motion Eq. (4) presenting exponential decay with distance

$$\mathbf{U}(kx_2)=\mathbf{a}e^{ikpx_2}, \quad \mathbf{t}(kx_2)=\mathbf{b}e^{ikpx_2}, \quad \mathcal{I}(p)>0,$$

where the constant vectors $\mathbf{a}$ and $\mathbf{b}$ are related through [12], p. 139: $b_i=(C_{k1i1}+pC_{i2k2})a_k$. Then the equations of motion Eq. (4) give

$$p\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}=\check{\mathbf{N}}\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \tag{6}$$

where $\check{\mathbf{N}}$ is the 4×4 matrix in Eq. (4). This eigenvalue problem yields a quartic for p. We limit our investigation to the subsonic range, defined as the greatest interval of values for v where the determinant of $\check{\mathbf{N}}-p\mathbf{1}$ possesses two roots p_1, p_2, with positive imaginary parts. We call $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{b}_1$, $\mathbf{b}_2$, the vectors $\mathbf{a}$ and $\mathbf{b}$ corresponding to each root. Then the solution is of the form ([12], p. 141)

$$\mathbf{U}=\mathbf{A}\langle e^{ikp^*}\rangle\mathbf{q}, \quad \mathbf{t}=\mathbf{B}\langle e^{ikp^*}\rangle\mathbf{q}, \quad \langle e^{ikp^*}\rangle=\mathrm{diag}(e^{ikp_1x_2},e^{ikp_2x_2}),$$

where $\mathbf{A}=[\mathbf{a}_1,\mathbf{a}_2]$, $\mathbf{B}=[\mathbf{b}_1,\mathbf{b}_2]$, and $\mathbf{q}$ is a constant vector. Using the boundary conditions Eq. (5), we have at the free surface $x_2=0$,

$$\mathbf{Bq}=0, \quad \text{and} \quad \mathbf{u}(x_1,0,x_3,t)=\mathbf{a}_{\mathrm{R}}e^{ik(x_1-vt)}, \quad \mathbf{a}_{\mathrm{R}}=\mathbf{Aq}. \tag{7}$$

Moreover, the matrices $\mathbf{A}$ and $\mathbf{B}$ satisfy the orthogonality condition, [23],

$$\overline{\mathbf{B}}^{\mathrm{T}}\mathbf{A}+\overline{\mathbf{A}}^{\mathrm{T}}\mathbf{B}=\mathbf{0}. \tag{8}$$

Now, the eigenrelation Eq. (6) may be generalized for any positive or negative integer n as

$$p^n\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}=\check{\mathbf{N}}^n\begin{bmatrix} \mathbf{a} \\ \mathbf{b} \end{bmatrix}, \quad \text{where } \check{\mathbf{N}}^n=\begin{bmatrix} \mathbf{N}_1^{(n)} & \mathbf{N}_2^{(n)} \\ \check{\mathbf{K}}^{(n)} & \mathbf{N}_1^{(n)\mathrm{T}} \end{bmatrix} \quad \text{(say)}. \tag{9}$$

Explicitly, the elements of $\check{\mathbf{N}}^n$ are computed by multiplication of $\check{\mathbf{N}}$ or its inverse by itself. For instance, $\check{\mathbf{K}}^{(n)}$ for $n=1$, 2, -1 is given by $\check{\mathbf{K}}^{(1)}=\check{\mathbf{N}}_3+X(1+\delta^2)\mathbf{1}$,

$$\begin{aligned} \check{K}_{11}^{(2)}&=-2s'_{16}[1-s'_{11}(1+\delta^2)X]/s'^2_{11}, \\ \check{K}_{12}^{(2)}&=[1-(s'_{11}-s'_{12})(1+\delta^2)X-2is'_{16}\delta X]/s'_{11},=\overline{\check{K}_{21}^{(2)}}, \\ \check{K}_{22}^{(2)}&=0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \check{K}_{11}^{(-1)}&=-[s'_{22}(1+\delta^2)-(s'_{11}s'_{22}-s'^2_{12})(1-\delta^2)^2X]X/D, \\ \check{K}_{12}^{(-1)}&=[s'_{26}(1+\delta^2)+2is'_{12}\delta+(s'_{12}s'_{16}-s'_{11}s'_{26})(1-\delta^2)^2X]X/D \\ &=\overline{\check{K}_{21}^{(-1)}}, \\ \check{K}_{22}^{(-1)}&=[1-(s'_{11}+s'_{66})(1+\delta^2)X+(s'_{11}s'_{66}-s'^2_{16})(1-\delta^2)^2X^2]/D, \end{aligned} \tag{11}$$

where D is a real denominator common to the $K_{ij}^{(-1)}$ whose expression is too long to reproduce and which turns out to be irrelevant for the derivation of the secular equation.

Now we write in turn the second vector line of Eq. $(9)_1$ for p_1 and for p_2, and deduce

$$\check{\mathbf{K}}^{(n)}\mathbf{A}+\check{\mathbf{N}}_1^{(n)}\mathbf{B}=\mathbf{B}\,\mathrm{diag}(p_1,p_2).$$

Multiplying this equality to the left by $\overline{\mathbf{a}_{\mathrm{R}}}^{\mathrm{T}}=\overline{\mathbf{q}}^{\mathrm{T}}\overline{\mathbf{A}}^{\mathrm{T}}$ and to the right by $\mathbf{q}$, and using Eqs. (7), (8), we conclude that (see [20] for the nonrotating case)

$$\overline{\mathbf{a}_{\mathrm{R}}}^{\mathrm{T}}\check{\mathbf{K}}^{(n)}\mathbf{a}_{\mathrm{R}}=0. \tag{12}$$

At $n=-1$, 1, 2, and $\mathbf{a}_{\mathrm{R}}=[1,\alpha]^{\mathrm{T}}$ (say), three equations follow:

$$\check{K}_{12}^{(-1)}\alpha+\overline{\check{K}_{12}^{(-1)}}\overline{\alpha}+\check{K}_{22}^{(-1)}\alpha\overline{\alpha}=-\check{K}_{11}^{(-1)},$$

$$\check{K}_{12}^{(1)}\alpha+\overline{\check{K}_{12}^{(1)}}\overline{\alpha}+\check{K}_{22}^{(1)}\alpha\overline{\alpha}=-\check{K}_{11}^{(1)},$$

$$\check{K}_{12}^{(2)}\alpha+\overline{\check{K}_{12}^{(2)}}\overline{\alpha}=-\check{K}_{11}^{(2)}.$$

We rearrange this system as: $F_{ik}g_k=h_i$, by introducing the following quantities:

$$F_{11}=D\mathcal{R}(\check{K}_{12}^{(-1)}), \quad F_{12}=D\mathcal{I}(\check{K}_{12}^{(-1)}), \quad F_{13}=D\check{K}_{22}^{(-1)},$$

$$F_{21}=0, \quad F_{22}=s'_{11}\mathcal{I}(\check{K}_{12}^{(1)}), \quad F_{23}=s'_{11}\check{K}_{22}^{(1)},$$

$$F_{31}=s'_{11}\mathcal{R}(\check{K}_{12}^{(2)}), \quad F_{32}=s'_{11}\mathcal{I}(\check{K}_{12}^{(2)}), \quad F_{33}=0,$$

$$g_1=\alpha+\overline{\alpha}, \quad g_2=i(\alpha-\overline{\alpha}), \quad g_3=\alpha\overline{\alpha},$$

$$h_1=-D\check{K}_{11}^{(-1)}, \quad h_2=-s'_{11}\check{K}_{11}^{(1)}, \quad h_3=-s'_{11}\check{K}_{11}^{(2)}.$$

Note that the explicit expressions for the nondimensional quantities F_{ik} and h_i in terms of $X=\rho v^2$, $\delta=\Omega/\omega$, and the s'_{ij} are easily read off Eqs. (10), (11). For instance, $F_{12}=2s'_{12}\delta X$, $F_{32}=-2s'_{16}\delta X$, $h_2=1-s'_{11}(1+\delta^2)X$, and so on.

The linear nonhomogeneous system $\mathbf{Fg}=\mathbf{h}$ has a unique solution for $\mathbf{g}$. Introducing $\Delta=\det\mathbf{F}$ and Δ_k $(k=1,2,3)$, the determinant of the matrix obtained from $\mathbf{F}$ by replacing its kth column with $\mathbf{h}$, we write the solution as $g_k=\Delta_k/\Delta$. But the components of $\mathbf{g}$ are related one to another through $g_3=(g_1/2)^2+(g_2/2)^2$. This relation is the *explicit secular equation for Rayleigh waves on an anisotropic crystal rotating in its plane of symmetry*,

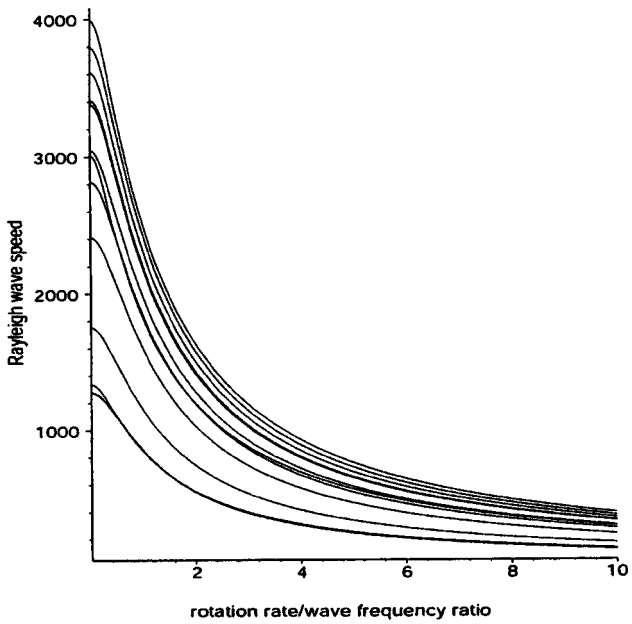

Fig. 2 Rayleigh wave speeds for 12 monoclinic crystals rotating about x_3

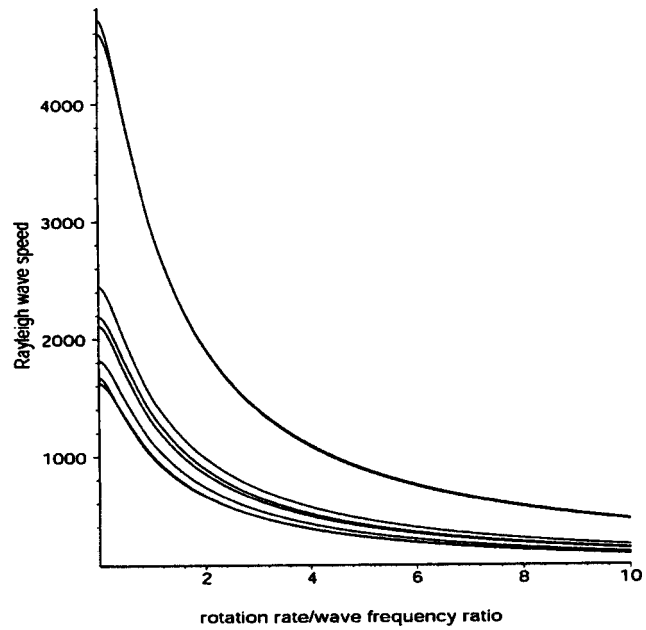

Fig. 3 Rayleigh wave speeds for 8 rhombic crystals rotating about x_3

$$\Delta_1^2+\Delta_2^2-4\Delta_3\Delta=0. \tag{13}$$

This equation is a polynomial of degree 8 in $X=\rho v^2$, and also of degree 8 in δ^2. Because $\delta=\Omega/\omega$ appears only in even powers in the secular equation, the Rayleigh speed obtained as a root of Eq. (13) does not depend on the sense of rotation. Numerically, we find that the rotation slows the Rayleigh wave down and that the speed is a monotone decreasing function of δ. We see this behavior on Fig. 2, where the dependence of the Rayleigh wave speed upon δ is shown for the 12 monoclinic crystals from Table 1. The curves are arranged in the same order as in the table, from the slowest (diphenyl, starting at 1276 m/s) to the fastest (diallage, starting at 4000 m/s).

The secular equation is valid for any crystal possessing at least one plane of symmetry, as long as the half-space is cut along a plane containing the normal to the plane of symmetry. In particular, it is also valid for orthorhombic crystals when the plane of cut contains one of the crystallographic axes. When this plane contains two crystallographic axes, the secular equation factorizes and a separate treatment is required.

4 Orthorhombic Materials

When the material possesses three orthogonal planes of symmetry and the axes $(O,\mathbf{i},\mathbf{j},\mathbf{k})$ are aligned with the crystallographic axes, some compliances vanish: $s'_{16}=s'_{26}=0$. Table 2 lists the values of the relevant reduced compliances for 8 rhombic crystals, computed from the corresponding values of the stiffnesses as collected by Shutilov [24]. The corresponding Rayleigh wave speed v_R in the nonrotating case (last column) is found from the exact secular equation, ([25])

$$(1-Xs'_{11})\sqrt{1-Xs'_{66}}-X\sqrt{s'_{11}[s'_{22}-X(s'_{11}s'_{22}-s'^2_{12})]}=0.$$

When the frame is rotating, the system $\mathbf{Fg}=\mathbf{h}$ reduces to

$$\begin{bmatrix} 0 & F_{12} & F_{13} \\ 0 & F_{22} & F_{23} \\ F_{31} & 0 & 0 \end{bmatrix}\begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}=\begin{bmatrix} h_1 \\ h_2 \\ 0 \end{bmatrix},$$

where $F_{31}\neq 0$ and

$$F_{12}=2s'_{12}\delta X,$$

$$F_{13}=1-(s'_{11}+s'_{66})(1+\delta^2)X+s'_{11}s'_{66}(1-\delta^2)^2X^2,$$

$$F_{22}=-2s'_{11}\delta X,\quad F_{23}=s'_{11}(1+\delta^2)X,$$

$$h_1=[s'_{22}(1+\delta^2)-(s'_{11}s'_{22}-s'^2_{12})(1-\delta^2)^2X]X,$$

$$h_2=1-s'_{11}(1+\delta^2)X.$$

From this new system of equations, we deduce that $g_2=\hat{\Delta}_2/\hat{\Delta}$ and $g_3=\hat{\Delta}_3/\hat{\Delta}$, where

$$\hat{\Delta}=F_{12}F_{23}-F_{22}F_{13},\quad \hat{\Delta}_2=h_1F_{23}-h_2F_{13},$$

$$\hat{\Delta}_3=F_{12}h_2-F_{22}h_1,$$

and also that $g_1=\alpha+\overline{\alpha}=0$, implying that $g_2=2i\alpha$, $g_3=-\alpha^2=(g_2/2)^2$ as well. This last equality is the *explicit secular equation for Rayleigh waves on an orthorhombic crystal rotating in one plane of symmetry*,

$$\hat{\Delta}_2^2-4\hat{\Delta}_3\hat{\Delta}=0.$$

This equation is a polynomial of degree 6 in $X=\rho v^2$ and in $\delta=\Omega/\omega$. As in the monoclinic case above, the roots are even functions of δ. Numerically, the results are similar to those of the monoclinic case, as Fig. 3 shows for the eight orthorhombic crys-

Table 2 Values of the reduced compliances (10^{-12} m^2/N), density (kg/m^3), and (nonrotating) surface wave speed (m/s) for 8 orthorhombic crystals

Material	s'_{11}	s'_{22}	s'_{12}	s'_{66}	ρ	v_R
sulfur	65.1	76.2	−42.2	132	2070	1628
iodic acid	36.1	20.1	−7.88	57.5	4630	1678
α-uranium	4.89	5.29	−1.13	13.5	19000	1819
rochelle salt	49.3	33.0	−18.2	102	1775	2114
sodium-tartrate	32.1	27.1	−16.8	102	1818	2197
strotium formate	24.5	30.9	−7.32	58.1	2250	2451
olivine	3.26	5.34	−0.97	12.6	3324	4599
benzophenone	13.0	13.9	−7.17	27.9	1219	4723

tals of Table 2. Again, the curves are arranged in the same order as in the table, from the slowest (sulfur, starting at 1628 m/s) to the fastest (benzophenone, starting at 4723 m/s).

5 Concluding Remarks

Several methods have been proposed to derive explicitly the secular equation for surface waves in nonrotating monoclinic crystals with the plane of symmetry at $x_3=0$. This author, [14], wrote the equations of motion as a system of two second-order differential equations for the tractions $\mathbf{t}$: $\hat{\alpha}_{ik}t_k''-i\hat{\beta}_{ik}t_k'-\hat{\gamma}_{ik}t_k=0$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ are 2×2 real symmetric matrices. Then the method of first integrals [26], yields the secular equation. The equations of motion Eq. (4) may also be written in a similar manner for a rotating crystal, but $\hat{\boldsymbol{\alpha}}$, $\hat{\boldsymbol{\beta}}$, $\hat{\boldsymbol{\gamma}}$ become complex and the method of first integrals is no longer applicable as such. Next, Ting [17] assumed an exponential form for $\mathbf{t}(kx_2)$ and obtained the secular equation through some simple algebraic manipulations, taking advantage of the fact that $\hat{\alpha}_{12}=\hat{\beta}_{22}=0$; in the rotating case, however, these quantities are no longer zero. Furs [27] (using the displacement field) and this author [25] (using the traction field) devised yet another method, where the secular equation is the resultant of two polynomials; again, having real quantities for the components of $\mathbf{N}$ is a crucial property, no longer true for rotating crystals.

All in all, it seems that the method of the polarization vector is the most appropriate for the case of a rotating crystal. Note that a simple derivation of its main result (12), not relying on the Stroh formalism, was presented recently, [21,22].

Acknowledgments

I thank T. C. T. Ting, G. Saccomandi, and A. Shuvalov for being so generous with their time and for helping me address some fundamental questions related to this material.

References

[1] Reindl, L., Scholl, G., Ostertag, T., Scherr, H., Wolff, U., and Schmidt, F., 1998, ''Theory and Application of Passive SWA Radio Transponders as Sensors,'' IEEE Trans. Ultrason. Ferroelectr. Freq. Control, **45**, pp. 1281–1292.

[2] Pohl, A., Ostermayer, G., Reindl, L., and Seifert, F., 1997, ''Monitoring the Tire Pressure at Cars Using Passive SWA Sensors,'' IEEE Ultrasonics Symposium, **1**, pp. 471–474.

[3] Pohl, A., Steindl, R., and Reindl, L., 1999, ''The 'Intelligent Tire' Utilizing Passive SWA Sensors—Measurement of Tire Friction,'' IEEE Trans. Instrum. Meas., **48**, pp. 1041–1046.

[4] Clarke, N. S., and Burdess, J. S., 1994, ''A Rotation Rate Sensor Based Upon a Rayleigh Resonator,'' ASME J. Appl. Mech., **61**, pp. 139–143.

[5] Clarke, N. S., and Burdess, J. S., 1994, ''Rayleigh Waves on a Rotating Surface,'' ASME J. Appl. Mech., **61**, pp. 724–726.

[6] Fang, H., Yang, J., and Jiang, Q., 2000, ''Rotation Perturbed Surface Acoustic Waves Propagating in Piezoelectric Crystals,'' Int. J. Solids Struct., **37**, pp. 4933–4947.

[7] Grigor'evskiĭ, V. I., Gulyaev, Yu. V., and Kozlov, A. I., 2000, ''Acoustic Waves in a Rotating Elastic Medium,'' Acoust. Phys., **46**, pp. 236–238.

[8] Collet, B., 2003, ''Gyroscopic Effect on Surface Acoustic Waves in Anisotropic Solid Media,'' *Proceedings of the 5th World Congress on Ultrasonics*, pp. 991–995.

[9] Jose, K. A., Suh, W. D., Xavier, P. B., Varadan, V. K., and Varadan, V. V., 2002, ''Surface Acoustic Wave MEMS Gyroscope,'' Wave Motion, **36**, pp. 367–381.

[10] Jahangir, E., and Howe, R. M., 1993, ''Time-Optimal Attitude Control Scheme for a Spinning Missile,'' J. Guid. Control Dyn., **16**, pp. 346–353.

[11] Schoenberg, M., and Censor, D., 1973, ''Elastic Waves in Rotating Media,'' Q. Appl. Math., **31**, pp. 115–125.

[12] Ting, T. C. T., 1996, *Anisotropic Elasticity: Theory and Applications*, Oxford University Press, New York.

[13] Chadwick, P., and Wilson, N. J., 1992, ''The Behavior of Elastic Surface Waves Polarized in a Plane of Material Symmetry, II. Monoclinic Media,'' Proc. R. Soc. London, Ser. A, **438**, pp. 207–223.

[14] Destrade, M., 2001, ''The Explicit Secular Equation for Surface Acoustic Waves in Monoclinic Elastic Crystals,'' J. Acoust. Soc. Am., **109**, pp. 1398–1402.

[15] Stroh, A. N., 1962, ''Steady State Problems in Anisotropic Elasticity,'' J. Math. Phys., **41**, pp. 77–103.

[16] Ingebrigsten, K. A., and Tonning, A., 1969, ''Elastic Surface Waves in Crystal,'' Phys. Rev., **184**, pp. 942–951.

[17] Ting, T. C. T., 2002, ''Explicit Secular Equations for Surface Waves in Monoclinic Materials With the Symmetry Plane at $x_1=0$, $x_2=0$ or $x_3=0$,'' Proc. R. Soc. London, Ser. A, **A458**, pp. 1017–1031.

[18] Currie, P. K., 1979, ''The Secular Equation for Rayleigh Waves on Elastic Crystals,'' Q. J. Mech. Appl. Math., **32**, pp. 163–173.

[19] Taziev, R. M., 1989, ''Dispersion Relation for Acoustic Waves in an Anisotropic Elastic Half-Space,'' Sov. Phys. Acoust., **35**, pp. 535–538.

[20] Ting, T. C. T., 2004, ''The Polarization Vector and Secular Equation for Surface Waves in an Anisotropic Elastic Half-Space,'' Int. J. Solids Struct., **41**, pp. 2065–2083.

[21] Destrade, M., 2004, ''Surface Waves in Rotating Rhombic Crystals,'' Proc. R. Soc. London, Ser. A, **460**, pp. 653–665.

[22] Destrade, M., 2004, ''Explicit Secular Equation for Scholte Waves Over a Monoclinic Crystal,'' J. Sound Vib., to appear.

[23] Barnett, D. M., and Lothe, J., 1973, ''Synthesis of the Sextic and the Integral Formalism for Dislocations, Green's Function, and Surface Wave (Rayleigh Wave) Solutions in Anisotropic Elastic Solids,'' Phys. Norv., **7**, pp. 13–19.

[24] Shutilov, V., 1988, *Fundamental Physics of Ultrasound*, Gordon and Breach, New York.

[25] Destrade, M., 2003, ''Rayleigh Waves in Symmetry Planes of Crystals: Explicit Secular Equations and Some Explicit Wave Speeds,'' Mech. Mater., **35**, 931–939.

[26] Mozhaev, V. G., 1995, ''Some New Ideas in the Theory of Surface Acoustic Waves in Anisotropic Media,'' *IUTAM Symposium on Anisotropy, Inhomogeneity and Nonlinearity in Solids*, D. F. Parker and A. H. England, eds., Kluwer, Dordrecht, The Netherlands, pp. 455–462.

[27] Furs, A. N., 1997, ''Covariant Form of the Dispersion Equation for Surface Acoustic Waves in Symmetry Planes of Crystals,'' Crystallogr. Rep., **4**, pp. 196–201.