Rayleigh Waves in Anisotropic Crystals Rotating About the Normal to a Symmetry Plane

The propagation of surface acoustic waves in a rotating anisotropic crystal is studied. The crystal is monoclinic and cut along a plane containing the normal to the symmetry plane; this normal is also the axis of rotation. The secular equation is obtained explicitly using the “method of the polarization vector,” and it shows that the wave is dispersive and decelerates with increasing rotation rate. The case of orthorhombic symmetry is also treated. The surface wave speed is computed for 12 monoclinic and 8 rhombohedral crystals, and for a large range of the rotation rate/wave frequency ratio.

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1 Introduction

Introduced more than 30 years ago, surface acoustic wave (SAW) devices have been used with great success by the telecommunication industry: nowadays, they are produced in large quantities (several billions per year) and used in wireless transmission and reception technology for color television sets, cell phones, global positioning systems, etc. In recent years, new applications for SAW devices have emerged, namely acoustic sensors, which are passive (no power supply is needed), resistant, almost nonaging, cheap (only one photo-lithographic process is involved in the production), light (less than 1g) and can be operated remotely and wirelessly. For instance [1], SAW identification tags are used for highway toll collection in Norway and for the Munich subway system; SAW temperature sensors can achieve a resolution of 0.02°C from −196°C up to 500°C; wide ranges and fine resolutions are also achieved for pressure, torque, or current sensors, etc. Also, the automotive industry is engaged in the search for an “intelligent tire” which could provide direct information on its current state as the car is moving; in this context SAW sensors have been used to measure tire pressure, [2], or friction, [3], as the wheel rotates. In general, SAW devices may be used as angular rate sensors (gyroscopes) to measure frequency shifts due to the rotation, [4–6]. In the present paper, an investigation of the effect of rotation upon the speed of surface (Rayleigh) waves in an anisotropic crystal is presented.

The crystal may possess as little as a single plane of symmetry. It is cut along any plane containing the normal to the symmetry plane and is assumed to rotate at a constant rate about this normal. The surface wave is polarized in the symmetry plane. In other words, it suffices to consider the propagation of a surface wave in the $x_1$-direction of a monoclinic crystal with symmetry plane at $x_3=0$, cut along the $x_2=0$ plane, and rotating about the $x_2$-axis (see Fig. 1). The secular equation for rotating materials was obtained by others but in simpler settings: by Clarke and Burdess in [4], or friction, [3], as the wheel rotates. In general, SAW devices may be used as angular rate sensors (gyroscopes) to measure frequency shifts due to the rotation, [4–6]. In the present paper, an investigation of the effect of rotation upon the speed of surface (Rayleigh) waves in an anisotropic crystal is presented.

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$$\sigma = \rho \ddot{u} + 2\mu (\nabla \nabla u + \nabla \nabla u^T) + p \Omega^2 \times \hat{k} \times (k \times u),$$

where $\sigma$ is the Cauchy stress tensor, $\rho$ is the constant mass density of the material, and the comma denotes differentiation. The sec-

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ond term in the right-hand side of Eq. (1) is due to the Coriolis acceleration, the third is due to the centrifugal acceleration. Note that Eq. (1) represents the time-dependent part of the full equations of motion. The time-independent part, namely \( \delta u' = \rho \Omega^2 k \times (k \times (u' + x)) \), where \( u' = u'(x) \) and \( \sigma_{ij}' = \epsilon_{ij} \mu'_{i j} \) must be solved separately. The questions remain of (a) whether an actual time-independent solution exists for all \( \Omega \) and if it does, of (b) whether the boundary conditions of a traction-free rotating half-space may be satisfied without perturbing the time-dependent boundary value problem. These questions do not seem to have been addressed in the literature, but some preliminary work seems however to suggest that (a) and (b) may be answered positively, at least within the framework of small amplitude waves superimposed upon a large elastic deformation.

Now, turning back to the time-dependent Eq. (1), the mechanical displacement \( u \) is taken in the form

\[
(u(x_1,x_2,x_3,t) = U(kx_2)e^{ik(x_1-\omega t)},
\]

(2)

showing a sinusoidal propagation with speed \( \omega \) and wave number \( k \) in the \( x_1 \)-direction, and the possibility of an attenuation in the \( x_2 \)-direction through the unknown function \( U(kx_2) \).

We wish to describe the influence of the frame rotation upon the speed of Rayleigh waves, and to this end, we introduce the following quantities:

\[
X = \rho \omega^2, \quad \delta = \Omega/(kv) = \Omega/\omega,
\]

where \( \omega \) is the real frequency of the wave.

For two-dimensional motions (\( \partial u/\partial x_3 = 0 \)) such as Eq. (2), the anisotropy of a crystal possessing \( x_3 = 0 \) as a symmetry plane is described by the following strain-stress relationship, \([12]\):

\[
\begin{bmatrix}
\varepsilon_{11}' & \varepsilon_{12}' & 0 & 0 & \varepsilon_{16}' \\
\varepsilon_{22}' & \varepsilon_{23}' & 0 & 0 & \varepsilon_{26}' \\
2\varepsilon_{21} & 2\varepsilon_{23} & \varepsilon_{44}' & \varepsilon_{45}' & 0 \\
2\varepsilon_{31} & 2\varepsilon_{32} & \varepsilon_{55}' & 0 & \varepsilon_{56}' \\
0 & 0 & \varepsilon_{66}' & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
\sigma_{11}' \\
\sigma_{22}' \\
\sigma_{23}' \\
\sigma_{31}' \\
\sigma_{32}' \\
\sigma_{12}'
\end{bmatrix},
\]

where the strain components \( \varepsilon_{ij}' \) are defined in terms of the displacement components by: \( \varepsilon_{ij}' = u_{ij}' + u_{ij} \), and the \( s_{ij}' \) are the reduced compliances. Alternatively, the equivalent strain-stress relations can be used, \([12]\, p. 39],

\[
\sigma' = C' \varepsilon', \quad C' S' = 1,
\]

(3)

where \( \sigma' = [\sigma_{11}, \sigma_{22}, \sigma_{23}, \sigma_{31}, \sigma_{32}] \), \( \varepsilon' = [\varepsilon_{11}, \ldots, 2\varepsilon_{12}]^T \). The \( C_{ij} \) are elements, in the Voigt notation, of the fourth-order elastic stiffness tensor \( C_{ijkl} \). Table 1 shows the relevant reduced compliances of 12 different monoclinic crystals, computed from the corresponding stiffnesses as collected by Chadwick and Wilson \([13]\); the last column gives the corresponding Rayleigh wave speed in the nonrotating case, \([14]\).

In view of the form Eq. (2) for the displacement, we introduce the functions \( t_1, t_2 \) for the tractions \( \sigma_{12}, \sigma_{22} \) on the planes \( x_2 = \pm \Omega/\omega \) as

\[
\begin{align*}
\sigma_{12}(x_1,x_2,x_3,t) & = it_1(t_2) e^{ik(x_1-\omega t)}, \\
\sigma_{22}(x_1,x_2,x_3,t) & = it_2(t_2) e^{ik(x_1-\omega t)}.
\end{align*}
\]

Then, substituting Eqs. (2) and (3) into the equations of motion (1), we derive the following system of linear first-order differential equations for \( U_1, U_2, t_1, t_2 \),

\[
\begin{bmatrix}
U' \\
\dot{U}'
\end{bmatrix} = \begin{bmatrix}
N_1 & N_2 \\
\dot{N}_1 + (1 - \delta^2)X1 & N_1
\end{bmatrix}
\begin{bmatrix}
U \\
t
\end{bmatrix},
\]

(4)

where \( U = [U_1,U_2]^T, t = [t_1,t_2]^T, \) and the prime denotes differentiation with respect to \( kx_2 \). Note that, as in the static case, \([15]\), the anisotropic strain (stress) decouples from the plane strain (stress) and need not be considered for this problem. This decoupling would not occur if the crystal was rotating about the \( x_1 \)-axis or the \( x_2 \)-axis, \([6]\).

The surface \( x_2 = 0 \) is free of tractions and so, the boundary conditions are

\[
t_1(0) = t_2(0) = 0.
\]

(5)

In Eq. (4), \( N_1 \) and \( N_2 \) are the same as the \( 2 \times 2 \) submatrices of the \( 6 \times 6 \) fundamental elasticity matrix \( N \) from Ingebrigsten and Tonning \([16]\). Their real matrix \( N_1 \), however, has been modified by the introduction of off-diagonal pure imaginary terms. Explicitly, we have

```

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<th>Material</th>
<th>( s_{11} )</th>
<th>( s_{12} )</th>
<th>( s_{16} )</th>
<th>( s_{16} )</th>
<th>( s_{16} )</th>
<th>( p )</th>
<th>( v_R )</th>
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```

Table 1. Values of the reduced compliances \((10^{-12} \text{ m}^2/\text{N})\), density \((\text{kg/m}^3)\), and (nonrotating) surface wave speed \((\text{m/s})\) for 12 monoclinic crystals

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\[-N_1 = \begin{bmatrix} r_6 & 1 \\ r_2 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} n_{66} & n_{26} \\ n_{26} & n_{22} \end{bmatrix}, \quad \tilde{N}_3 = \begin{bmatrix} \eta & 2i\delta X \\ -2i\delta X & 0 \end{bmatrix},\]

where the quantities \( r_6, r_2, r_{63}, r_{23}, n_{26}, n_{66}, \eta \) are given in terms of the elastic parameters as \([14,17]\)

\[
\eta = \frac{1}{s_{11}^{(1)}}, \quad r_6 = -\frac{s_{16}^{(1)}}{s_{11}^{(1)}}, \quad r_2 = -\frac{s_{12}^{(1)}}{s_{11}^{(1)}}, \quad n_{26} = \frac{1}{s_{11}^{(1)} s_{12}^{(1)} s_{22}^{(1)}}, \quad n_{66} = \frac{1}{s_{11}^{(1)} s_{16}^{(1)} s_{66}^{(1)}}.
\]

Thus the rotation of the crystal perturbs the equations of motion in three ways: the introduction of dispersion through \( \delta \) a shift of magnitude \( \delta X \) in the lower left submatrix of \( N \) proportional to the \( 2 \times 2 \) unit matrix; and the modification of \( n_{26} \), which is diagonal in the nonrotating case (note that the new matrix \( \tilde{N}_3 \) is Hermitian: \( \tilde{N}_3 = \tilde{N}_3^\dagger \)). Despite these modifications, the secular equation can be obtained explicitly for the surface wave speed, using a method proposed by Currie [18] and by Taziev [19] for nonrotating anisotropic crystals with and without a plane of symmetry, respectively.

### 3 Secular Equation

The method of the polarization vector was first presented by Currie [18] to derive the secular equation for Rayleigh waves in the symmetry plane of monoclinic nonrotating crystals. Then Taziev [19] generalized the method to triclinic (no symmetry plane) crystals. This method takes advantage of the Cayley-Hamilton theorem for the fundamental matrix \( N \), which implies that only \( n \) matrices \( N^k \) \((k = 1, \ldots, n)\) are linearly independent \((n = 3 \) for monoclinic crystals, \( n = 5 \) for triclinic crystals). Currie used the matrices \( N, N^2, N^3 \); Taziev, the matrices \( N, N^2, N^3, N^4, N^5 \). Recently, Ting [20] placed their results within the context of the Stroh-Barnett-Lothe formalism and improved on them by showing that the choices of \( N^{-1}, N, N^3 \) for monoclinic crystals and of \( N^{-2}, N^{-1}, N, N^2, N^3 \) for triclinic crystals lead to simpler and more explicit secular equations. His approach is now adapted to our present context of a rotating crystal with one symmetry plane. An alternative derivation, not based on the Stroh-Barnett-Lothe formalism, is available elsewhere.\[21,22\].

We seek solutions to the equations of motion Eq. (4) presenting exponential decay with distance

\[
U(kx_2) = a e^{i k p x_2}, \quad T(kx_2) = b e^{i k p x_2}, \quad (T \neq 0),
\]

where the constant vectors \( a \) and \( b \) are related through \([12]\), p. 139: \( b = (C_{111} + p C_{122}) a \). Then the equations of motion Eq. (4) give

\[
\begin{pmatrix} a \\ b \end{pmatrix} = \tilde{N} \begin{pmatrix} a \\ b \end{pmatrix},
\]

(6)

where \( \tilde{N} \) is the \( 4 \times 4 \) matrix in Eq. (4). This eigenvalue problem yields a quartic for \( p \). We limit our investigation to the subspace range, defined as the greatest interval of values for \( v \) where the determinant of \( \tilde{N} - pI \) possesses two roots \( p_1, p_2 \), with positive imaginary parts. We call \( a_1, a_2, b_1, b_2 \), the vectors \( a \) and \( b \) corresponding to each root. Then the solution is of the form \([12]\), p. 141)

\[
U = A(e^{i k p x}) \quad a = B(e^{i k p x}) \quad (e^{i k p x}) = \text{diag}(e^{i k p x_1}, e^{i k p x_2}),
\]

where \( A = [a_1, a_2] \), \( B = [b_1, b_2] \), and \( q \) is a constant vector. Using the boundary conditions Eq. (5), we have at the free surface \( x_2 = 0 \)

\[
\begin{pmatrix} a \\ b \end{pmatrix} = 0,
\]

and \( u(x_1,0,x_3,t) = a_8 e^{i (k_{11} - 1) \eta t} \), \( a_8 = Aq \).

Moreover, the matrices \( A \) and \( B \) satisfy the orthogonality condition, \([23]\).
\[ \Delta_1^2 + \Delta_2^2 - 4\Delta_3 \Delta = 0. \] (13)

This equation is a polynomial of degree 8 in \( X = \rho v^2 \), and also of degree 8 in \( \delta^2 \). Because \( \delta = \Omega/\omega \) appears only in even powers in the secular equation, the Rayleigh speed obtained as a root of Eq. (13) does not depend on the sense of rotation. Numerically, we find that the rotation slows the Rayleigh wave down and that the speed is a monotone decreasing function of \( \delta \). We see this behavior on Fig. 2, where the dependence of the Rayleigh wave speed upon \( \delta \) is shown for the 12 monoclinic crystals from Table 1. The curves are arranged in the same order as in the table, from the slowest (diphenyl, starting at 1276 m/s) to the fastest (diallage, starting at 4000 m/s).

The secular equation is valid for any crystal possessing at least one plane of symmetry, as long as the half-space is cut along a plane containing the normal to the plane of symmetry. In particular, it is also valid for orthorhombic crystals when the plane of cut contains one of the crystallographic axes. When this plane contains two crystallographic axes, the secular equation factorizes and a separate treatment is required.

4 Orthorhombic Materials

When the material possesses three orthogonal planes of symmetry and the axes \((O,i,j,k)\) are aligned with the crystallographic axes, some compliances vanish: \( s''_{16} = s''_{26} = 0 \). Table 2 lists the values of the relevant reduced compliances for 8 rhombic crystals, computed from the corresponding values of the stiffnesses as collected by Shutilov [24]. The corresponding Rayleigh wave speed \( v_R \) in the nonrotating case (last column) is found from the exact secular equation, (125)

\[ (1 - X s''_{11}) \sqrt{1 - X s''_{66} - X \sqrt{s''_{11} s''_{22} - X(s''_{11} s''_{22} - s''_{12}^2)}} = 0. \]

When the frame is rotating, the system \( \mathbf{F} = h \) reduces to

\[
\begin{bmatrix}
0 & F_{12} & F_{13} \\
0 & F_{22} & F_{23} \\
F_{31} & 0 & 0
\end{bmatrix}
\begin{bmatrix}
g_1 \\
g_2 \\
g_3
\end{bmatrix}
= \begin{bmatrix}
h_1 \\
h_2 \\
h_3
\end{bmatrix},
\]

where \( F_{31} \neq 0 \) and

\[
F_{13} = 2 s_{12} \delta X, \\
F_{13} = 1 - (s''_{11} + s''_{66}) (1 + \delta^2) X + s''_{11} s''_{66} (1 - \delta^2)^2 X^2, \\
F_{22} = -2 s_{12} \delta X, \\
h_1 = [s''_{22} (1 + \delta^2) - (s''_{11} s''_{22} - s''_{12}^2) (1 - \delta^2)^2 X], \\
h_2 = 1 - s''_{11} (1 + \delta^2) X.
\]

From this new system of equations, we deduce that \( g_2 = \hat{\Delta}_2 / \hat{\Delta} \) and \( g_3 = \hat{\Delta}_3 / \hat{\Delta} \), where

\[
\hat{\Delta} = F_{12} F_{23} - F_{13} F_{22}, \\
\hat{\Delta}_2 = h_1 F_{23} - h_2 F_{13}, \\
\hat{\Delta}_3 = F_{12} h_2 - F_{22} h_1,
\]

and also that \( g_1 = \alpha + \bar{\alpha} = 0 \), implying that \( g_2 = 2i\alpha \), \( g_3 = -\alpha^2 = (g_2/2)^2 \) as well. This last equality is the explicit secular equation for Rayleigh waves on an orthorhombic crystal rotating in one plane of symmetry,

\[ \Delta_2^2 - 4\Delta_3 \Delta = 0. \]

This equation is a polynomial of degree 6 in \( X = \rho v^2 \) and in \( \delta = \Omega/\omega \). As in the monoclinic case above, the roots are even functions of \( \delta \). Numerically, the results are similar to those of the monoclinic case, as Fig. 3 shows for the eight orthorhombic crys-

![Fig. 2 Rayleigh wave speeds for 12 monoclinic crystals rotating about x3](image1)

![Fig. 3 Rayleigh wave speeds for 8 rhombic crystals rotating about x3](image2)
tals of Table 2. Again, the curves are arranged in the same order as in the table, from the slowest (sulfur, starting at 1628 m/s) to the fastest (benzophenone, starting at 4723 m/s).

5 Concluding Remarks

Several methods have been proposed to derive explicitly the secular equation for surface waves in nonrotating monoclinic crystals with the plane of symmetry at $\theta = 0$. This author, [14], wrote the equations of motion as a system of two second-order differential equations for the tractions $\hat{t}_k$, $\hat{t}_k^i$ and $\hat{t}_k^j$, where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ are $2 \times 2$ real symmetric matrices. Then the method of first integrals [26], yields the secular equation. The equations of motion Eq. (4) may also be written in a similar manner for a rotating crystal, but $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$ become complex and the method of first integrals is no longer applicable as such. Next, Ting [17] assumed an exponential form for $t(kx_2)$ and obtained the secular equation through some simple algebraic manipulations, taking advantage of the fact that $\hat{\alpha}_{12} = \hat{\alpha}_{22} = 0$; in the rotating case, however, these quantities are no longer zero. Furs [27] (using the displacement field) and this author [25] (using the traction field) devised yet another method, where the secular equation is the resultant of two polynomials; again, having real quantities for the components of $N$ is a crucial property, no longer true for rotating crystals.

All in all, it seems that the method of the polarization vector is the most appropriate for the case of a rotating crystal. Note that a formalism, was presented recently, using the traction field and including non-rotating results as a limiting case of the rotating crystal. This method, together with the original secular equation method, is a crucial property, no longer true for rotating crystals.

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References