**CanCNSM**
Victoria
June 16-20, 1999

# FINITE–AMPLITUDE INHOMOGENEOUS PLANE WAVES IN A DEFORMED BLATZ-KO MATERIAL

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**Key Words:** Solid Blatz–Ko material, Finite amplitude, Inhomogeneous plane waves, Exact static solutions, Energy propagation

**Abstract.** *The possibility of having finite–amplitude inhomogeneous plane waves propagating in a compressible hyperelastic isotropic material maintained in a state of arbitrary finite static homogeneous deformation is investigated.*

*The material considered belongs to a special class of Blatz-Ko materials, describing the behaviour of a certain solid polyurethane elastomer. The waves are of complex exponential type and are linearly–polarized in a direction orthogonal to both the direction of propagation and the direction of attenuation.*

*The propagation of energy is considered, and well–known results established in the linearized theory are recovered, despite the nonlinearity of the motion.*

*As by–products of the study, new static inhomogeneous solutions are found.*

## 1 INTRODUCTION

The Blatz–Ko models for polyurethane rubber have been extensively used to describe the behaviour of compressible hyperelastic isotropic materials undergoing finite deformations. Good agreement with experiments has been obtained in many measurements.[1] Their elastic response functions depend only on $\det \mathbb{F}$, where $\mathbb{F}$ is the deformation gradient.

The experimental and theoretical bases for these models have been laid down by Blatz & Ko[1] and later further justified by Beatty & Stalnaker[2] and Knowles & Sternberg.[3] In these papers, two subclasses of Blatz–Ko materials were derived, namely the 'Blatz–Ko foamed, polyurethane elastomer' and the 'Blatz–Ko solid, polyurethane rubber'. For the foamed Blatz–Ko material, the strain–energy function is independent of $I = \text{tr}\,\mathbb{B}$, whilst for the solid Blatz–Ko material, it is independent of $II = [(\text{tr}\,\mathbb{B})^2 - \text{tr}\,\mathbb{B}^2]/2$, where $\mathbb{B} = \mathbb{F}\mathbb{F}^{\text{T}}$ is the left Cauchy–Green strain tensor. Following the works of Knowles & Sternberg,[3–5] Horgan[6] recently proved that the model for the Blatz–Ko solid, polyurethane rubber was the only one of the general Blatz–Ko class for which ellipticity was to hold when arbitrary large three–dimensional stretches were applied to the material.

Carroll[7] proved that circularly–polarized finite–amplitude plane waves may propagate in a biaxially deformed homogeneous isotropic compressible material. Later, Boulanger, Hayes & Trimarco[8] extensively studied similar waves in a triaxially deformed Hadamard material. Lately, vibrations and stability of finitely deformed compressible materials have received attention from various authors (see Roxburgh & Ogden,[9] Vandyke & Wineman,[10] Akyuz & Ertepinar[11]). However, these articles either deal with homogeneous plane waves or are placed within the framework of small deformations superimposed on large .

In the present paper, we study the propagation of inhomogeneous waves, in particular linearly–polarized transverse finite–amplitude inhomogeneous plane waves in a solid Blatz–Ko material, when it is maintained in a state of static homogeneous deformation.

The form of the paper is as follows. In Section 2, we recall how the solid Blatz–Ko material is defined. We assume that this material is first maintained in a state of homogeneous deformation and then, that a finite motion of complex exponential type is superposed.

Although linearly–polarized transverse finite–amplitude inhomogeneous plane waves may not propagate in a general compressible material in the absence of body forces,[12] we show in Section 3 that such motions are possible for the class of solid Blatz–Ko materials, provided that the directions of the normals to the planes of constant phase and of constant amplitude are conjugate with respect to the ellipsoid $\mathbf{x}.\mathbb{B}\mathbf{x} = 1$, where $\mathbb{B}$ is the left Cauchy–Green strain tensor corresponding to the primary homogeneous deformation.

In Section 4, the propagation of the energy carried by the waves is considered. Despite the non–linearity introduced by the finite magnitude of the waves, we recover energetics results established earlier in the linearized theory. If $\mathbf{p}$ and $\mathbf{q}$ are unit normals to the planes of constant amplitude and of constant phase, respectively, then we prove, inter–

alia, that $\check{\mathbf{R}}.\mathbf{p} = 0$ and $\check{\mathbf{R}}.\mathbf{q} = v\check{\overline{E}}$. Here, $\check{\mathbf{R}}$ and $\check{\overline{E}}$ are the respective time averages of the energy flux vector and total energy and $v$ is the speed of propagation of the wave.

Also of interest is the question of possible static finite deformations of the Blatz–Ko material. Ever since Ericksen[13] proved that only homogeneous deformations were possible for every compressible, isotropic, elastic solid (in the absence of body forces), inhomogeneous deformations have been sought for restricted classes of compressible materials. The interest of such non–universal solutions was emphasized by Currie & Hayes.[14] However, with regard to finite deformations possible for a solid Blatz–Ko material, few solutions have been proposed. We cite the works of Knowles[15] for anti–plane shear (see also Agarwal,[16] Erterpinar & Eraslanoglu,[17] Poliglone & Horgan[18]). In Section 5, we find new exact static solutions by letting the speed of propagation of the finite–amplitude inhomogeneous wave tend to zero. This means that a deformation consisting of a finite static exponential type displacement, superposed upon a finite static triaxial stretch is possible for the Blatz–Ko solid, polyurethane rubber, in the absence of body forces.

## 2 BASIC EQUATIONS

### 2.1 Primary static deformation

For a solid Blatz–Ko material, undergoing a deformation characterized by a deformation gradient $\mathbb{F}$, the strain–energy function $\Sigma$ is given by

$$\Sigma = \mu(\operatorname{tr}\mathbb{F}\mathbb{F}^{\mathrm{T}} - 3)/2 + \kappa f(J). \tag{1}$$

Here, $J = \det\mathbb{F}$, $\mu$ and $\kappa$ are constants (the shear and bulk moduli for infinitesimal deformations) and $f$ a function of $J$ alone. Extensive discussions on the validity of this model and explicit expressions for $f$ can be found in,[1,2] for instance. Note also that Chadwick & Jarvis[19] refer to materials with energy function given by (1) as 'restricted Hadamard materials', because the corresponding expression for a Hadamard material is:[20] $\Sigma = \mu(\operatorname{tr}\mathbb{B} - 3)/2 - \nu[(\operatorname{tr}\mathbb{B})^2 - (\operatorname{tr}\mathbb{B}^2)]/2 + \kappa f(J)$, where $\mathbb{B} = \mathbb{F}\mathbb{F}^{\mathrm{T}}$, and $\nu$ is a constant.

From (1) follows that for solid Blatz–Ko materials, the Cauchy stress tensor $\mathbb{T}$ is given by

$$\mathbb{T} = \kappa f'(J)\mathbf{1} + \mu J^{-1}\mathbb{B}. \tag{2}$$

Suppose that the material is first subjected to a pure homogeneous static deformation, with extension ratios $\lambda_1$, $\lambda_2$, and $\lambda_3$ along the principal directions given by the orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$, respectively. Thus, a material particle initially at position $\mathbf{X} = X_i\mathbf{e}_i$ has moved to the position $\mathbf{x} = x_i\mathbf{e}_i$ where $x_i = \lambda_i X_i$ (no sum), in the static state of pure homogeneous deformation. The corresponding deformation gradient $\mathbb{F}$ and left Cauchy–Green strain tensor $\mathbb{B}$ are given by

$$\mathbb{F} = \operatorname{diag}(\lambda_1, \lambda_2, \lambda_3), \quad \mathbb{B} = \operatorname{diag}(\lambda_1^2, \lambda_2^2, \lambda_3^2), \quad \text{with } J = \det\mathbb{F} = \lambda_1\lambda_2\lambda_3. \tag{3}$$

### 2.2 Superposed deformation

We now superpose a linearly–polarized inhomogeneous plane wave of finite amplitude upon the large static deformation. Let **a** be a unit vector in the direction of polarization. We assume that a particle at $\mathbf{x}$ in the intermediate static state has moved to $\overline{\mathbf{x}}$ such that

$$\overline{\mathbf{x}} = \mathbf{x} + \alpha\{e^{i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}\mathbf{a} = \mathbf{x} + 2\alpha e^{-\omega\mathbf{S}^-.\mathbf{x}}\cos\omega(\mathbf{S}^+.\mathbf{x}-t)\mathbf{a}. \tag{4}$$

Here, $\alpha$ is a finite real scalar, $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$ is a complex vector, called the 'slowness bivector',[21] $\omega$ is the real frequency and 'c.c.' denotes the complex conjugate. The planes defined by $\mathbf{S}^+.\mathbf{x} = const.$ are the planes of constant phase and those defined by $\mathbf{S}^-.\mathbf{x} = const.$ are the planes of constant amplitude.

We seek linearly–polarized plane wave solutions to the equations of motion which are 'transverse', in the sense that the direction of polarization is orthogonal to both directions of propagation and of attenuation, i. e. $\mathbf{a}.\mathbf{S} = 0$.

The deformation gradient $\overline{\mathbb{F}}$ associated with the motion (4) is given by

$$\overline{\mathbb{F}} = \partial\overline{\mathbf{x}}/\partial\mathbf{X} = \widetilde{\mathbb{F}}\mathbb{F}, \quad \text{where} \quad \widetilde{\mathbb{F}} = \mathbf{1} + \alpha\mathbf{a}\otimes\{i\omega e^{i\omega(\mathbf{S}.\mathbf{x}-t)}\mathbf{S} + \text{c.c.}\}. \tag{5}$$

The corresponding left Cauchy–Green tensor is $\overline{\mathbb{B}}$ given by $\overline{\mathbb{B}} = \overline{\mathbb{F}}\,\overline{\mathbb{F}}^{\mathrm{T}} = \widetilde{\mathbb{F}}\mathbb{B}\widetilde{\mathbb{F}}^{\mathrm{T}}$.

Using $\mathbf{a}.\mathbf{S} = 0$ and (5), we find that the determinants $\widetilde{J}$ of $\widetilde{\mathbb{F}}$ and $\overline{J}$ of $\overline{\mathbb{F}}$ are given by

$$\widetilde{J} = 1, \quad \text{and} \quad \overline{J} = \widetilde{J}\,J = J = \lambda_1\lambda_2\lambda_3. \tag{6}$$

Hence, the Cauchy stress $\overline{\mathbb{T}}$ associated with the motion (4) is given by

$$\overline{\mathbb{T}} = \kappa f'(J)\mathbf{1} + \mu J^{-1}\overline{\mathbb{B}} = \kappa f'(J)\mathbf{1} + \mu J^{-1}\widetilde{\mathbb{F}}\mathbb{B}\widetilde{\mathbb{F}}^{\mathrm{T}}. \tag{7}$$

## 3 PROPAGATING SOLUTIONS

### 3.1 Equations of motion

In the absence of body forces, the equations of motion read

$$\mathbf{div}\,\overline{\mathbb{T}} = \overline{\rho}\frac{\partial^2\overline{\mathbf{x}}}{\partial t^2}, \quad \frac{\partial\overline{\mathbb{T}}_{ij}}{\partial\overline{x}_j} = \overline{\rho}\frac{\partial^2\overline{\mathbf{x}}_i}{\partial t^2}. \tag{8}$$

Here $\overline{\rho}$ is the mass density per unit volume of the solid Blatz–Ko material, measured in the current state of static deformation. It is related to $\rho_0$, the mass density per unit volume of the body in its undeformed state, through $\overline{\rho} = J^{-1}\rho_0$. Upon using the divergence theorem, we can easily prove that (8) is equivalent to

$$\mathbf{div}\,(\widetilde{J}\,\overline{\mathbb{T}}\widetilde{\mathbb{F}}^{-\mathrm{T}}) = \overline{\rho}\widetilde{J}\,\frac{\partial^2\overline{\mathbf{x}}}{\partial t^2}, \quad \frac{\partial(\widetilde{J}\,\overline{\mathbb{T}}\widetilde{\mathbb{F}}^{-\mathrm{T}})_{ij}}{\partial x_j} = \overline{\rho}\widetilde{J}\,\frac{\partial^2\overline{\mathbf{x}}_i}{\partial t^2} \tag{9}$$

With (6), (7), and the equation $\partial(\widetilde{J}\,\widetilde{\mathbb{F}}_{ij}^{-\mathrm{T}})/\partial x_j = 0$, the equations of motion reduce to

$$\mu\{(\mathbf{S}.\mathbb{B}\mathbf{S})e^{i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}\mathbf{a} = \rho_0\{e^{i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}\mathbf{a}. \tag{10}$$

### 3.2 Propagating evanescent waves

From equation (10) we deduce the following propagation condition,

$$\mathbf{S}.\mathbb{B}\mathbf{S} = c_0^{-2}, \quad \text{with} \quad c_0 = \sqrt{\mu/\rho_0}. \tag{11}$$

Hence, $\mathbf{S}.\mathbb{B}\mathbf{S}$ is real, which is equivalent to $\mathbf{S}^+.\mathbb{B}\mathbf{S}^- = 0$. In other words, the directions of the normal to the planes of constant phase and of the normal to the planes of constant amplitude are conjugate with respect to the $\mathbb{B}$–ellipsoid.

We now solve the propagation condition. For homogeneous waves with displacement in the form $\{e^{i(\mathbf{k}.\mathbf{x}-vt)} + \text{c.c.}\}\mathbf{a}$ where $\mathbf{k}$ is a real vector, the direction of propagation (direction of $\mathbf{k}$) is chosen and the wave speed $v$ and polarization direction (direction of $\mathbf{a}$) are then found by solving an eigenvalue problem. For inhomogeneous waves, the direction of propagation (that of $\mathbf{S}^+$) does not coincide with the direction of attenuation (that of $\mathbf{S}^-$), so that it is actually the plane containing these two directions (the plane of the ellipse of the bivector $\mathbf{S}$) which is typically prescribed. Solving the equations of motion then yields $\mathbf{a}$, $\mathbf{S}^+$ and $\mathbf{S}^-$ which gives the directions of polarization, propagation and attenuation, the angle between the planes of constant phase and of constant amplitude, the wave speed ($\mid \mathbf{S}^+ \mid^{-1}$) and attenuation factor ($\omega \mid \mathbf{S}^- \mid$) . This can be done by using the 'Directional Ellipse Method' introduced by Hayes.[21]

The Directional Ellipse of $\mathbf{S}$ is by definition the ellipse similar and similarly situated to the ellipse of $\mathbf{S}$ whose minor semi–axis is of unit length. Thus, calling $m$ ($m \geq 1$) the length of the major semi–axis, this ellipse is that of the bivector $\mathbf{C}$ defined by

$$\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}, \tag{12}$$

where $\hat{\mathbf{m}}$ and $\hat{\mathbf{n}}$ are orthogonal unit vectors along the semi–axes of the ellipse of $\mathbf{S}$. The bivector $\mathbf{S}$ may now be written as

$$\mathbf{S} = N\mathbf{C}, \quad N = Te^{i\phi}, \tag{13}$$

where $N$ is a complex number of modulus $T$ and argument $\phi$.

Once $\mathbf{C}$ is prescribed, knowledge of $T$ and $\phi$ gives all the information needed about the wave. Indeed, we have,[21]

$$\begin{cases} \mathbf{S}^+ = T(m\cos\phi\hat{\mathbf{m}} - \sin\phi\hat{\mathbf{n}}), & \mid \mathbf{S}^+ \mid = T\sqrt{m^2\cos^2\phi + \sin^2\phi}, \\ \mathbf{S}^- = T(m\sin\phi\hat{\mathbf{m}} + \cos\phi\hat{\mathbf{n}}), & \mid \mathbf{S}^- \mid = T\sqrt{m^2\sin^2\phi + \cos^2\phi}. \end{cases} \tag{14}$$

The angle between the planes of constant phase and of constant amplitude is $\theta$, given by

$$\tan\theta = 2m/[(m^2-1)\sin 2\phi]. \tag{15}$$

The wave is linearly–polarized along $\mathbf{a} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$.

In our case, we have, according to (11),

$$N^{-2} = T^{-2}e^{-2i\phi} = c_0^2\mathbf{C}.\mathbb{B}\mathbf{C}. \tag{16}$$

Note that $N$ becomes infinite when $\mathbf{C}.\mathbb{B}\mathbf{C} = 0$. This case will be considered later (see Section 5). Provided that $\mathbf{C}.\mathbb{B}\mathbf{C} \neq 0$, then $T$ and $\phi$ are given by

$$\begin{cases} T^{-2} = c_0^2\sqrt{(m^2\hat{\mathbf{m}}.\mathbb{B}\hat{\mathbf{m}} - \hat{\mathbf{n}}.\mathbb{B}\hat{\mathbf{n}})^2 + 4m^2(\hat{\mathbf{m}}.\mathbb{B}\hat{\mathbf{n}})^2}, \\ \tan 2\phi = -2m(\hat{\mathbf{m}}.\mathbb{B}\hat{\mathbf{n}})/(m^2\hat{\mathbf{m}}.\mathbb{B}\hat{\mathbf{m}} - \hat{\mathbf{n}}.\mathbb{B}\hat{\mathbf{n}}). \end{cases} \tag{17}$$

For illustrative purposes, we construct the following example. We let $\lambda_1 = 1$, $\lambda_2 = \sqrt{2}$ and $\lambda_3 = \sqrt{3}$, so that $\mathbb{B} = \text{diag}\,(1, 2, 3)$, and we prescribe $\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}$ as follows:

$$m = (1 + \sqrt{3})/2, \quad \hat{\mathbf{m}} = (-\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}, \quad \hat{\mathbf{n}} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}. \tag{18}$$

With this choice, $T$ and $\phi$ are given by $T^2 = c_0^2\sqrt{3}/(4 + 4\sqrt{3})$, and $\phi = 5\pi/12$.

Therefore, the following motion is possible for a solid Blatz–Ko material:

$$\overline{\mathbf{x}} = \mathbf{x} + 2\alpha\mathbf{a}e^{-\omega\mathbf{S}^-.\mathbf{x}}\cos\omega(\mathbf{S}^+.\mathbf{x} - t), \quad \text{where} \quad \mathbf{x} = \text{diag}\,(1, \sqrt{2}, \sqrt{3})\mathbf{X}. \tag{19}$$

This wave is linearly–polarized along $\mathbf{a}$, given by

$$\mathbf{a} = (\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}. \tag{20}$$

It propagates in the direction of $\mathbf{S}^+$, with speed $v =\mid \mathbf{S}^+ \mid^{-1}$, given by

$$\mathbf{S}^+ = -c_0^{-1}(1 - 1/\sqrt{3})^{1/2}[(1 + 2\sqrt{3})\mathbf{e}_1 + (1 + \sqrt{3})\mathbf{e}_2 + \mathbf{e}_3]/8, \quad v = 4c_0/\sqrt{3}. \tag{21}$$

It is attenuated in the direction of $\mathbf{S}^-$, with attenuation factor $\sigma = \omega \mid \mathbf{S}^- \mid$, given by

$$\mathbf{S}^- = -c_0^{-1}(1 - 1/\sqrt{3})^{1/2}[(4 + \sqrt{3})\mathbf{e}_1 + (1 - \sqrt{3})\mathbf{e}_2 - (2 + 3\sqrt{3})\mathbf{e}_3]/8, \quad \sigma = 3\omega/(4c_0). \tag{22}$$

Finally, the angle between the planes of constant phase and amplitude is $\theta$, such that $\tan\theta = -2\sqrt{2}$.

### 3.3 Bounds for the velocity

We recall the propagation condition,

$$\mathbf{S}^+.\mathbb{B}\mathbf{S}^+ - \mathbf{S}^-.\mathbb{B}\mathbf{S}^- - = \rho_0/\mu. \tag{23}$$

We now introduce the speed of propagation $v$ for the finite–amplitude plane wave, defined by $v =\mid \mathbf{S}^+ \mid^{-1}$, and the unit vector $\mathbf{p}$ in the direction of propagation, defined by $\mathbf{p} = \mathbf{S}^+/\mid \mathbf{S}^+ \mid$. Dividing (23) by $\mathbf{S}^+.\mathbf{S}^+$ yields

$$\rho_0 v^2 = \mu(\mathbf{p}.\mathbb{B}\mathbf{p} - v^2\mathbf{S}^-.\mathbb{B}\mathbf{S}^-),$$

or

$$\rho_0 v^2 = \mu \frac{\mathbf{p}.\mathbb{B}\mathbf{p}}{1 + (\mu/\rho_0)\mathbf{S}^-.\mathbb{B}\mathbf{S}^-}, \tag{24}$$

The numerator of the fraction in (24) is bounded bellow by $\lambda_1^2$ (propagation in the direction of least stretch) and above by $\lambda_3^2$ (propagation in the direction of greatest stretch), whilst the denominator is bounded bellow by 1 (when $\mid \mathbf{S}^- \mid \rightarrow 0$, and the wave is homogeneous), and not bounded above (when $\mid \mathbf{S}^- \mid \rightarrow \infty$, and the amplitude of the wave tends to zero).

We conclude that the greatest velocity for the finite–amplitude plane wave of exponential type is $v_{\text{max}}$, given by

$$\rho_0 v_{\text{max}}^2 = \mu\lambda_3^2,$$

and it is attained when the wave propagates with no attenuation, in the direction of greatest stretch. This result was first established by Hayes [QJMAM 1968] for small–amplitude homogeneous plane waves propagating in a deformed restricted Hadamard material. Hayes also proved that waves of this type travel with the least speed $c_{\text{min}}$ in the direction of least stretch, with $c_{\text{min}}$ given by $\rho_0 c_{\text{min}}^2 = \mu\lambda_1^2$. In contrast with homogeneous waves, we see that inhomogeneous waves have a lower bound $v_{\text{min}}$ given by

$$\rho_0 v_{\text{min}}^2 = 0.$$

We will show in Section 5 how it is possible to construct such finite inhomogeneous static deformations.

## 4 ENERGY PROPAGATION

Now, we consider the energy carried by the waves. We introduce two energy quantities, the total energy density, sum of the kinetic and stored–energy densities, and the energy flux vector, which measures the flux of energy crossing a unit surface per unit time.

Following the works of Schouten[23] and Synge,[24] Hayes[25] showed that for an inhomogeneous plane wave with slowness bivector $\mathbf{S} = \mathbf{S}^+ + i\mathbf{S}^-$, we have $\check{\mathbf{R}}.\mathbf{S}^- = 0$ and $\check{\mathbf{R}}.\mathbf{S}^+ = \check{E}$, where $\check{\mathbf{R}}$ and $\check{E}$ are the temporal mean values of the energy flux vector and total energy density carried by the wave, respectively. This result holds for *any* linear conservative system, whether or not the medium is anisotropic or subject to an internal constraint, such as incompressibility or inextensibility.

The above mentioned papers are situated within the linearized theory. Here, we find similar results in the non–linear case of a finite–amplitude wave propagating in a finitely deformed solid Blatz–Ko material.

We introduce the following notation to denote temporal mean values: if $D(\mathbf{x}, t)$ is a periodic field quantity with frequency $\omega$, then its mean value is $\check{D}$ defined by

$$\check{D} = \frac{\omega}{2\pi} \int_0^{\frac{\omega}{2\pi}} D(\mathbf{x}, t) dt. \tag{25}$$

We begin with the kinetic energy density per unit volume $\overline{K}$, measured in the current configuration,

$$\overline{K} = \rho(\dot{\mathbf{x}}.\dot{\mathbf{x}})/2. \tag{26}$$

The velocity $\dot{\mathbf{x}}$ is derived by differentiating (4) with respect to time $t$, so that

$$\overline{K} = -\rho\alpha^2\omega^2\{e^{2i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}/2 + \rho\alpha^2\omega^2 e^{-2\omega\mathbf{S}^-.\mathbf{x}}, \tag{27}$$

and the mean kinetic energy density $\check{\overline{K}}$ is

$$\check{\overline{K}} = \rho\alpha^2\omega^2 e^{-2\omega\mathbf{S}^-.\mathbf{x}} = \rho_0 J^{-1}\alpha^2\omega^2 e^{-2\omega\mathbf{S}^-.\mathbf{x}}. \tag{28}$$

According to (11), we may rewrite this last equality as

$$\check{\overline{K}} = \mu J^{-1}\alpha^2\omega^2 (\mathbf{S}.\mathbb{B}\mathbf{S}) e^{-2\omega\mathbf{S}^-.\mathbf{x}}. \tag{29}$$

The stored–energy density $\overline{W}$ associated with the wave, and measured per unit volume in the *reference* configuration, is defined by

$$\overline{W} = \overline{\Sigma} - \Sigma = \mu(\overline{I} - I)/2 + \kappa[f(\overline{J}) - f(J)], \tag{30}$$

where $\Sigma$ and $\overline{\Sigma}$ are the strain–energy functions corresponding to the static deformation $x_i = \lambda_i X_i$ (no sum) and to the motion (4), respectively.

In (30), $\overline{I}$ and $I$ are the respective first principal invariants of $\overline{\mathbb{B}}$ and $\mathbb{B}$. Taking the trace of $\overline{\mathbb{B}} = \widetilde{\mathbb{F}}\mathbb{B}\widetilde{\mathbb{F}}^{\text{T}}$, yields

$$\overline{I} = I - \alpha^2\omega^2\{(\mathbf{S}.\mathbb{B}\mathbf{S})e^{2i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\} + 2\alpha^2\omega^2(\mathbf{S}.\mathbb{B}\mathbf{S}^*)e^{-2\omega\mathbf{S}^-.\mathbf{x}}, \tag{31}$$

where $\mathbf{S}^* = \mathbf{S}^+ - i\mathbf{S}^-$. Using (31) and also (6), we write the mean value $\check{\overline{W}}$ of the stored–energy density as

$$\check{\overline{W}} = \mu\,\alpha^2\omega^2(\mathbf{S}.\mathbb{B}\mathbf{S}^*)e^{-2\omega\mathbf{S}^-.\mathbf{x}}. \tag{32}$$

We are now able to compute the total energy density $\overline{E}$, in the *current* configuration, as: $\overline{E} = \overline{K} + J^{-1}\overline{W}$. Using (29) and (32), we can write directly the mean value of $\overline{E}$ as

$$\check{\overline{E}} = 2\mu J^{-1}\alpha^2\omega^2(\mathbf{S}.\mathbb{B}\mathbf{S}^+)e^{-2\omega\mathbf{S}^-.\mathbf{x}}. \tag{33}$$

Now we turn our attention to the energy flux vector, $\overline{\mathbf{R}}$ (say). By definition it is such that its component $\overline{\mathbf{R}}.\mathbf{p}$ (where $\mathbf{p}$ is a unit vector) is equal to the rate at which the total energy $\overline{E}$ crosses, at current time, unit area of surface having outward normal $\mathbf{p}$ in the final state of deformation. In the context of elasticity, $\overline{\mathbf{R}} = -\overline{\mathbb{T}}.\dot{\overline{\mathbf{x}}}$. It is related to the energy flux vector $\mathbf{R}$ (say), measured in the intermediate static state of deformation through[22]

$$\mathbf{R} = \widetilde{J}\widetilde{\mathbb{F}}^{-1}\overline{\mathbf{R}} = -\widetilde{\mathbb{F}}^{-1}\overline{\mathbb{T}}.\dot{\overline{\mathbf{x}}} = \alpha\{i\omega e^{i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}\widetilde{\mathbb{F}}^{-1}\overline{\mathbb{T}}\mathbf{a}. \tag{34}$$

It is found that $\mathbf{R}$ is given by

$$\mathbf{R} = \alpha\{i\omega e^{i\omega(\mathbf{S}.\mathbf{x}-t)} + \text{c.c.}\}[\mathbb{T}\mathbf{a} + \mu J^{-1}\alpha\{i\omega e^{i\omega(\mathbf{S}.\mathbf{x}-t)}\mathbb{B}\mathbf{S} + \text{c.c.}\}]. \tag{35}$$

The temporal mean value $\check{\mathbf{R}}$ of the energy flux vector $\mathbf{R}$ is therefore

$$\check{\mathbf{R}} = 2\mu J^{-1}\alpha^2\omega^2 e^{-2\omega\mathbf{S}^-.\mathbf{x}}\,\mathbb{B}\mathbf{S}^+. \tag{36}$$

Hence, using equation (33), we see that $\check{\mathbf{R}}.\mathbf{S} = \check{\overline{E}}$.

From this last equation we conclude first that

$$\check{\mathbf{R}}.\mathbf{S}^- = 0, \quad \text{or} \quad \check{\mathbf{R}}.\mathbf{p} = 0, \tag{37}$$

where $\mathbf{p}$ is the unit normal to the planes of constant amplitude, and second that

$$\check{\mathbf{R}}.\mathbf{S}^+ = \check{\overline{E}}, \quad \text{or} \quad \check{\mathbf{R}}.\mathbf{q} = v\check{\overline{E}} \tag{38}$$

where $\mathbf{q}$ is the unit normal to the planes of constant phase and $v$ is the speed of the wave, given by $v = \mid \mathbf{S}^+ \mid^{-1}$.

## 5 STATIC EXPONENTIAL SOLUTIONS

Finally, we exploit an interesting feature of complex exponential solutions. For propagating solutions, the displacement is the real part of the complex quantity $\alpha\mathbf{a}e^{i\omega N(\mathbf{C}.\mathbf{x}-N^{-1}t)}$. However, because $\mathbf{C}$ may be arbitrarily prescribed, it may happen that for certain choices, $N^{-1} = 0$, thus removing the time dependence of the solution. The product $\omega N = k$(say) may nevertheless remain finite and the static deformation $\alpha\mathbf{a}\{e^{ik\mathbf{C}.\mathbf{x}} + \text{c.c.}\}$ may be superimposed upon the pure homogeneous deformation. Hence, exact solutions to the equations of motion are found. Such solutions have been called 'Static Exponential Solutions' (SES) by Boulanger & Hayes -.[26]

In our context, we mentioned in Section 3.2 that for certain choices of $\mathbf{C}$, $N^{-1}$ is equal to zero. This case arises when the bivector $\mathbf{C} = m\hat{\mathbf{m}} + i\hat{\mathbf{n}}$ is prescribed to be such that

$$\mathbf{C}.\mathbb{B}\mathbf{C} = 0, \quad \text{or} \quad m^2\hat{\mathbf{m}}\mathbb{B}\hat{\mathbf{m}} - \hat{\mathbf{n}}\mathbb{B}\hat{\mathbf{n}} = \hat{\mathbf{m}}\mathbb{B}\hat{\mathbf{n}} = 0, \tag{39}$$

which means that the ellipse of $\mathbf{C}$ is similar and similarly situated to the elliptical section of the $\mathbb{B}$–ellipsoid by the plane of $\mathbf{C}$.[27] Note that there is an infinity of such bivectors $\mathbf{C}$.

In conclusion, we have proved that the deformation transporting a material point from $\mathbf{X} = X_i\mathbf{e}_i$ to the position $\overline{\mathbf{x}}$, given by

$$\overline{\mathbf{x}} = \mathbf{x} + 2\alpha\mathbf{a}e^{-k\hat{\mathbf{n}}.\mathbf{x}}\cos mk(\hat{\mathbf{m}}.\mathbf{x}), \tag{40}$$

where $x_i = \lambda_i X_i$ (no sum), is possible for a solid Blatz–Ko material, in the absence of body forces. In (40), the real number $m$ and the two orthogonal unit vectors $\hat{\mathbf{m}}$, $\hat{\mathbf{n}}$ are such that $m^2\hat{\mathbf{m}}\mathbb{B}\hat{\mathbf{m}} - \hat{\mathbf{n}}\mathbb{B}\hat{\mathbf{n}} = \hat{\mathbf{m}}\mathbb{B}\hat{\mathbf{n}} = 0$, $\alpha$ and $k$ are arbitrary real numbers, and $\mathbf{a} = \hat{\mathbf{m}} \times \hat{\mathbf{n}}$.

As an example, we consider the simple case where $\hat{\mathbf{m}}$ is in a principal plane ($\hat{\mathbf{m}} = \cos\phi\,\mathbf{e}_1 + \sin\phi\,\mathbf{e}_3$), and $\hat{\mathbf{n}}$ in a principal direction ($\hat{\mathbf{n}} = \mathbf{e}_2$). By choosing $m$ as follows,

$$m^2 = \lambda_2^2/(\lambda_1^2\cos^2\phi + \lambda_3^2\sin^2\phi), \tag{41}$$

we can construct an inhomogeneous deformation of a solid Blatz-Ko material, which may be written as

$$\begin{cases} \overline{x} = \lambda_1 X - 2\alpha\sin\phi\, e^{-k\lambda_2 Y}\cos mk(\lambda_1 X\cos\phi + \lambda_3 Z\sin\phi), \\ \overline{y} = \lambda_2 Y, \\ \overline{z} = \lambda_3 Z + 2\alpha\cos\phi\, e^{-k\lambda_2 Y}\cos mk(\lambda_1 X\cos\phi + \lambda_3 Z\sin\phi), \end{cases} \tag{42}$$

where $k$ is arbitrary and $m$ is given by (41).

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