WAVES AND VIBRATIONS IN A SOLID OF SECOND GRADE

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We study the viscoelastic second grade solid, for which the constitutive equation consists in the sum of a purely elastic part and a viscoelastic part; this latter part is specified by two microstructural coefficients α_1 and α_2, in addition to the usual Newtonian viscosity ν. We show via some exact solutions that such solids may describe some interesting dispersive effects. The solutions under investigation belong to special classes of standing waves and of circularly-polarized finite-amplitude waves.

1. Introduction

Fosdick and Yu[1] studied the thermodynamics and the non-linear oscillations of a special class of of viscoelastic solids of differential type where the rate effects are characterized by two microstructural coefficients α_1 and α_2, in addition to the usual Newtonian viscosity ν. This is a most promising model, which generalizes the usual dissipative solid used in many applications (for example in non-linear acoustics[2]) and which introduces more than one characteristic time or speed. This model is the solid-mechanic analogue of the much-discussed second grade fluid and for this reason it has been dubbed "the solid of second grade". Here we point out some important dynamic features of this solid by considering some simple, finite-amplitude, motions.

In Section 3, we record some unexpected behaviors for the incompressible Mooney-Rivlin solid of second grade, for which the elastic part of the Cauchy stress tensor is linear in the first two invariants of the Cauchy-Green tensor. We use a finite-amplitude, rectilinear shear motion to study the influence of rate effects on some classical problems. First, we find that the creep and recovery experiments will undergo some time-dependent oscillations if those effects are strong enough (in the corresponding purely elastic case, creep and recovery have exponential time dependence.) Second, we describe a slab with one face fixed and the other oscillating: we find that, above a certain threshold, the rate effects will cause the amplitude of the resulting oscillations within the slab to be rapidly (exponentially) attenuated with distance from the moving plate, *even when there is no Newtonian viscosity* (in the corresponding purely elastic case, the oscillations are transmitted sinusoidally through the slab.) We argue that these effects are intimately linked to the microstructure of the solid via the parameter α_1, and that they are symptomatic of the behavior of solids of second grade in general and are not restricted to these two particular problems, nor to the specialization to the Mooney-Rivlin model.

This point is further pressed in Section 4, where we make a connection with the results of a previous article, for finite-amplitude, elliptically polarized, transverse plane waves. We find that the nature of generalized oscillatory shearing motions[3] might not change with the introduction of second grade effects, but that the nature of sinusoidal standing waves is affected at high frequencies.

2. Basic equations

As is conventional in continuum mechanics, the motion of a body is described here by a relation $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$, where $\mathbf{x}$ denotes the current coordinates at time t of a point occupied by the particle of coordinates $\mathbf{X}$ in the reference configuration. The deformation gradient $\mathbf{F}(\mathbf{X}, t)$ and the spatial velocity gradient $\mathbf{L}(\mathbf{X}, t)$ associated with the motion are defined by $\mathbf{F} := \partial\mathbf{x}/\partial\mathbf{X}$ and $\mathbf{L} := \text{grad}\ \dot{\mathbf{x}}$, respectively (here the dot denotes the material derivative, see Chadwick[4] for instance). The other geometrical and kinematical quantities of interest are the left Cauchy-Green strain tensor $\mathbf{B} := \mathbf{F}\mathbf{F}^T$ and the first two Rivlin-Ericksen tensors $\mathbf{A}_1 := \mathbf{L} + \mathbf{L}^T$ and $\mathbf{A}_2 := \dot{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{L} + \mathbf{L}^T\mathbf{A}_1$.

In this article we restrict our attention to incompressible materials so that only isochoric motions are possible; hence in particular, det $\mathbf{B} = 1$ and tr $\mathbf{D} = 0$ at all times.

Further, we consider incompressible materials with a special constitutive equation[1], for which the Cauchy stress $\mathbf{T}$ is split in an additive manner into an *elastic part* $\mathbf{T}^{\mathrm{E}}$ and a *dissipative part* $\mathbf{T}^{\mathrm{D}}$. For the elastic part, the constitutive equation is that of a general hyperelastic, incompressible, isotropic material, that is[4]

$$\mathbf{T}^{\mathrm{E}} = -p\mathbf{1} + 2\frac{\partial \Sigma}{\partial I_1}\mathbf{B} - 2\frac{\partial \Sigma}{\partial I_2}\mathbf{B}^{-1}, \tag{1}$$

where $\mathbf{1}$ is the unit tensor and $p = p(\mathbf{x}, t)$ is the arbitrary Lagrange multiplier associated with the constraint of incompressibility and is usually called the pressure. In Eq. (1), Σ is the strain energy function, which for an incompressible material depends on I_1 and I_2, the first two principal invariants of $\mathbf{B}$: $I_1 :=$ tr $\mathbf{B}$, $I_2 := [I_1^2 - \text{tr } (\mathbf{B}^2)]/2$. The dissipative part is given by

$$\mathbf{T}^{\mathrm{D}} = \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2, \tag{2}$$

where ν is the usual classical Newtonian viscosity and α_1, α_2 are the microstructural coefficients. The total Cauchy stress is the sum of these two parts, namely

$$\mathbf{T} = -p\mathbf{1} + 2\frac{\partial \Sigma}{\partial I_1}\mathbf{B} - 2\frac{\partial \Sigma}{\partial I_2}\mathbf{B}^{-1} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{3}$$

Note that for the Mooney-Rivlin model of rubber elasticity, for which $2\Sigma = C(I_1 - 3) + E(I_2 - 3)$, where C and E are material constants, the total stress specializes to

$$\mathbf{T} = -p\mathbf{1} + C\mathbf{B} - E\mathbf{B}^{-1} + \nu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2. \tag{4}$$

For materials which such a constitutive equation, Hayes and Saccomandi[5] studied the propagation of finite amplitude transverse waves superimposed onto an arbitrary homogeneous static deformation.

3. On the nature of the microstructural parameter α_1

Fosdick and Yu[1] showed that the compatibility of the material model Eq. (3) with the laws of thermodynamics requires that $\alpha_1 + \alpha_2 = 0$, and that the Newtonian viscosity ν is non-negative. Moreover, they showed that the Helmholtz free energy function of the second grade solid is a minimum at equilibrium if and only if $\alpha_1 \geq 0$. Another interesting and important result of their investigation is a remark about the evolution in time of $E(t)$, the *canonical energy* of a body mechanically isolated and contained in a region Ω_t free of tractions. The canonical energy is the sum of the Helmholtz free energy and the kinetic energy and so it is defined by

$$E(t) := \int_{\Omega_t} \left(\Sigma + \tfrac{1}{4}\alpha_1 \mathbf{A}_1 \cdot \mathbf{A}_1\right) \mathrm{d}v + \int_{\Omega_t} \tfrac{1}{2}\rho\dot{\mathbf{x}} \cdot \dot{\mathbf{x}} \mathrm{d}v, \tag{5}$$

(where ρ is the mass density). Fosdick and Yu showed that its time evolution is given by

$$\dot{E} = -\frac{\nu}{2} \int_{\Omega_t} (\mathbf{A}_1 \cdot \mathbf{A}_1)\, \mathrm{d}v, \tag{6}$$

which means that the canonical energy decreases with time. This latter equation also shows that α_1 does not contribute to dissipation. This is a central point in our discussion.

Now the aim of this Section is to elaborate further on the parameter α_1 and to show that it is genuinely connected with the microstructure of the material (precisely, with a characteristic length). To this end we try to maintain algebraic complexity at the lower level and so we study the following simple rectilinear shear motion,

$$\mathbf{x}(\mathbf{X}, t) = [X + u(Y, t)]\, \mathbf{e}_1 + Y\mathbf{e}_2 + Z\mathbf{e}_3, \tag{7}$$

where u is a yet unknown function and $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ form a fixed orthonormal triad.

For this isochoric motion, the kinematical quantities of interest are

$$\begin{aligned}
&\mathbf{B} = \mathbf{1} + u_Y^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + u_Y(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),\\
&\mathbf{B}^{-1} = \mathbf{1} + u_Y^2 \mathbf{e}_2 \otimes \mathbf{e}_2 - u_Y(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),\\
&\mathbf{A}_1 = u_{Yt}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1),\\
&\mathbf{A}_2 = u_{Ytt}(\mathbf{e}_1 \otimes \mathbf{e}_2 + \mathbf{e}_2 \otimes \mathbf{e}_1) + 2u_{Yt}^2 \mathbf{e}_2 \otimes \mathbf{e}_2,\\
&\mathbf{A}_1^2 = u_{Yt}^2(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2),
\end{aligned} \tag{8}$$

and the first two principal invariants are equal: $I_1 = I_2 = 3 + u_Y^2$.

The equations of motions, in the absence of body forces, are: div $\mathbf{T} = \rho\ddot{\mathbf{x}}$ in current form. Writing them in referential form, and using Eqs. (8) which, together with Eq. (3), show that the non-diagonal elements of $\mathbf{T}$ depend on Y and t only, we find that they reduce to

$$-\frac{\partial p}{\partial X} + \frac{\partial T_{12}}{\partial Y} = \rho u_{tt}, \quad \frac{\partial T_{22}}{\partial Y} = 0, \quad -\frac{\partial p}{\partial Z} = 0. \tag{9}$$

According to the third of these equations, $p = p(X, Y, t)$. Then, by differentiation of the first and second equations in Eqs. (9) with respect to X, we find that $p_{XX} = p_{YX} = 0$, so that $p_X = p_1(t)$, say. Finally, the first equation in Eqs. (9) is the determining equation for the shearing deformation, given by

$$-p_1(t) + (Qu_Y)_Y + \nu u_{YYt} + \alpha_1 u_{YYtt} = \rho u_{tt}, \tag{10}$$

where $Q = Q(u_Y^2) = 2(\partial\Sigma/\partial I_1 + \partial\Sigma/\partial I_2)$ is the *generalized shear modulus*[3]. To simplify further the computations in the remainder of this Section, we take the scalar $p_1(t)$ to be identically zero ($p_1 \equiv 0$), and the constitutive equation of the material to be Eq. (4). In that case, $Q = \text{const.} = \mu$, where $\mu = C + E$ is the infinitesimal shear modulus of Mooney-Rivlin materials, and the determining equation Eq. (10) simplifies to

$$\mu u_{YY} + \nu u_{YYt} + \alpha_1 u_{YYtt} = \rho u_{tt}. \tag{11}$$

We notice that this partial differential equation is exactly the same equation as that obtained by Rubin et al.[6] in the linear limit of the phenomenological theory of dispersion caused by an inherent material characteristic length; it also occurs in the framework of acoustic waves propagation in bubbly liquids[7]. If in Eq. (11) L is a characteristic length and T a characteristic time, then we may exhibit three dimensionless numbers: $\pi_1 = \nu/(\mu T)$, $\pi_2 = \alpha_1/(\mu T^2)$ and $\pi_3 = \rho L/(\mu T^2)$.

We now pause to examine the behavior of our Mooney-Rivlin second grade solid when it undergoes some classic experiments of viscoelasticity, namely the *creep* experiment and the *recovery* experiment. These are *quasi-static* experiments, so that the inertial term on the right hand-side of Eq. (11) may be neglected; in other words, $\pi_3 \ll 1$ here. Then, integrating with respect to Y gives

$$\mu u_Y + \nu u_{Yt} + \alpha_1 u_{Ytt} = \text{const.}, \tag{12}$$

the constant being dictated by the type of experiment to be modeled. For creep and for recovery, the shearing deformation is considered homogeneous so that here, u is of the form $u(Y,t) = K(t)Y$. For recovery, $K \to 0$ as $t \to \infty$ and for creep, K tend to a finite value, K^0 say. Hence the governing equations are

$$K + K_\tau + \varepsilon K_{\tau\tau} = K^0, \quad K + K_\tau + \varepsilon K_{\tau\tau} = 0, \tag{13}$$

for creep and for recovery, respectively. Here, Eq. (12) has been non-dimensionalized by the scaling $\tau = \mu t/\nu$ (note that ν/μ is usually called the relaxation time) and by the introduction of the dimensionless parameter $\varepsilon = \mu\alpha_1/\nu^2$. When $\varepsilon = 0$, we recover the classical solutions

$K(\tau) = K^0(1 - \mathrm{e}^{-\tau})$ and $K(\tau) = K^0\mathrm{e}^{-\tau}$ for the creep problem and for the recovery problem, respectively, where K^0 is the steady state amount of shear. When $\varepsilon \leq 1/4$, the general solutions of Eq. (13) are also damped, but when when $\varepsilon > 1/4$, the parameter α_1 $(> 4\nu^2/\mu)$ modifies the nature of the solutions. Hence for the recovery experiment, the solution is of the form

$$K(\tau) = K^0\mathrm{e}^{-\tau/2\varepsilon}\cos(\sqrt{4\varepsilon - 1}\,\tau/2\varepsilon), \tag{14}$$

clearly highlighting that microstructural oscillations come into play.

Now turning back to the *dynamic* equation Eq. (11), we consider a slab of thickness L in the Y direction, and we introduce the dimensionless variables $\xi := Y/L$ and $\tau := \sqrt{\mu/(\rho L^2)}\,t$. Also, we consider the case where $\nu \equiv 0$ (no Newtonian viscosity); then Eq. (11) is recast as

$$u_{\xi\xi} + \frac{\alpha_1}{\rho L^2}u_{\xi\xi\tau\tau} = u_{\tau\tau}, \tag{15}$$

where we note that the quantity $\alpha_1/(\rho L^2)$ is a dimensionless parameter. Let the slab be sandwiched between two rigid plates, one at the bottom $\xi = 0$, oscillating with frequency ω and displacement $U\cos\omega t$ (say) and one at the top $\xi = 1$, at rest. We seek an exact solution to Eq. (15) in separable form, $u = U\phi(\xi)\cos(\omega t)$, say. Then ϕ satisfies

$$\left(1 - \frac{\alpha_1\omega^2}{\mu}\right)\phi'' + \frac{\rho L^2\omega^2}{\mu}\phi = 0, \quad \phi(0) = 1, \quad \phi(1) = 0. \tag{16}$$

When $\alpha_1 = 0$, we recover the classical elastic case solution,

$$\phi(\xi) = \frac{\sin\sqrt{\rho/\mu}\,L\omega(1-\xi)}{\sin\sqrt{\rho/\mu}\,L\omega}. \tag{17}$$

When $\alpha_1\omega^2/\mu < 1$, the solution is essentially of the same nature:

$$\phi(\xi) = \frac{\sin\alpha(1-\xi)}{\sin\alpha}, \quad \text{with} \quad \alpha := \frac{\sqrt{\rho/\mu}\,L\omega}{\sqrt{1 - (\alpha_1/\mu)\,\omega^2}}, \tag{18}$$

and so, at "low" frequencies there is no fundamental difference between the purely elastic case and the Mooney-Rivlin second grade solid. When $\alpha_1\omega^2/\mu > 1$ however, the microstructure (via the parameter α_1) dramatically alters the response of the viscoelastic slab, because then the solution is in the form

$$\phi(\xi) = \frac{\sinh\beta(1-\xi)}{\sinh\beta}, \quad \text{with} \quad \beta := \frac{\sqrt{\rho/\mu}\,L\omega}{\sqrt{(\alpha_1/\mu)\,\omega^2 - 1}}. \tag{19}$$

Hence at "high" frequencies, the vibrations engendered in the slab by the lower plate are localized near the vibrating plate; the oscillations are "trapped" in the microstructure. Figure 1 shows the variations of the oscillations's amplitude through the thickness of the slab. Figure 1(a) represents $\phi(\xi)$ given by Eq. (18) for $\alpha = 5, 9, 23$; these vibrations are sinusoidal, with resonance occurring at $\alpha = n\pi$. Figure 1(b) represents $\phi(\xi)$ given by Eq. (19) for $\beta = 5, 9, 23$; the amplitude is rapidly attenuated through the thickness; note that as $\alpha_1\omega^2/\mu$ approaches 1 from above, the quantity β tends to infinity, and the localization is greater and greater until at $\alpha_1\omega^2/\mu = 1$, we have perfect isolation. These attenuation effects can exist in viscoelastic solids for which $\nu \neq 0$, $\alpha_1 = 0$, but at the expense of canonical energy dissipation. Here we emphasize that the results are obtained for a Mooney-Rivlin solid of second grade with *no Newtonian viscosity*, $\nu = 0$, so that by Eq. (6), *the canonical energy is conserved*.

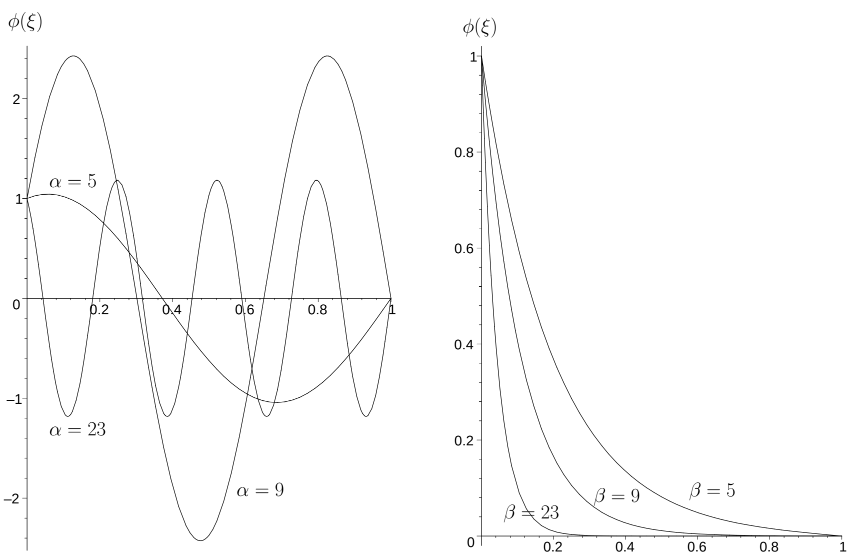

Figure 1. Influence of the microstructure on the oscillations of a slab fixed on one end. At low frequencies: oscillations; at high frequencies: attenuation.

4. Finite amplitude transverse plane waves

In this Section we consider the following class of motions

$$x = \gamma X + u(z,t), \quad y = \gamma Y + v(z,t), \quad z = \lambda Z, \tag{20}$$

that is, motions describing a transverse wave, polarized in the (XY) plane, and propagating in the Z direction of a solid subject to a pure homogeneous equi-biaxial pre-stretch along the X, Y, and Z axes, with corresponding constant principal stretch ratios γ, γ, and λ ($\gamma^2\lambda = 1$). Here u and v are yet unknown scalar functions. Then the geometrical quantities of interest are

$$\mathbf{B} = \begin{bmatrix} \gamma^2 + \lambda^2 u_z^2 & \lambda^2 u_z v_z & \lambda^2 u_z \\ \lambda^2 u_z v_z & \gamma^2 + \lambda^2 v_z^2 & \lambda^2 v_z \\ \lambda^2 u_z & \lambda^2 v_z & \lambda^2 \end{bmatrix}, \quad \mathbf{B}^{-1} = \begin{bmatrix} \lambda & 0 & -\lambda u_z \\ 0 & \lambda & -\lambda v_z \\ -\lambda u_z & -\lambda v_z & \lambda(u_z^2 + v_z^2) + \gamma^4 \end{bmatrix}, \tag{21}$$

and the kinematical quantities of interest are

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 0 & u_{zt} \\ 0 & 0 & v_{zt} \\ u_{zt} & v_{zt} & 0 \end{bmatrix}, \quad \mathbf{A}_1^2 = \begin{bmatrix} u_{zt}^2 & u_{zt} v_{zt} & 0 \\ u_{zt} v_{zt} & v_{zt}^2 & 0 \\ 0 & 0 & u_{zt}^2 + v_{zt}^2 \end{bmatrix}, \tag{22}$$

and

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & u_{ztt} \\ 0 & 0 & v_{ztt} \\ u_{ztt} & v_{ztt} & 2(u_{zt}^2 + v_{zt}^2) \end{bmatrix}, \tag{23}$$

and the first two invariants are

$$I_1 = 2\gamma^2 + \lambda^2 + \lambda^2(u_z^2 + v_z^2), \quad I_2 = 2\lambda + \gamma^4 + \lambda(u_z^2 + v_z^2). \tag{24}$$

Now the equations of motion, in the absence of body forces, are given in current form as div $\mathbf{T} = \rho \ddot{\mathbf{x}}$, or here,

$$-\frac{\partial p}{\partial x} + \frac{\partial T_{13}}{\partial z} = \rho u_{tt}, \quad -\frac{\partial p}{\partial y} + \frac{\partial T_{23}}{\partial z} = \rho v_{tt}, \quad \frac{\partial T_{33}}{\partial z} = 0. \tag{25}$$

Differentiating these equations with respect to x, we find $p_{xx} = p_{yx} = p_{zx} = 0$, so that $p_x = p_1(t)$, say. Similarly, by differentiating the equations with respect to y, we find $p_y = p_2(t)$, say. Now the first two equations reduce to

$$\begin{aligned} &- p_1(t) + (Qu_z)_z + \nu u_{zzt} + \alpha_1 u_{zztt} = \rho u_{tt}, \\ &- p_2(t) + (Qv_z)_z + \nu v_{zzt} + \alpha_1 v_{zztt} = \rho v_{tt}, \end{aligned} \tag{26}$$

and the third equation determines p. Here, $Q = Q(u_z^2 + v_z^2)$ is the generalized shear modulus, now defined by

$$Q = 2(\lambda^2 \partial\Sigma/\partial I_1 + \lambda\partial\Sigma/\partial I_2). \tag{27}$$

Following Destrade and Saccomandi[8], we take the derivative of Eqs. (26) with respect z, we introduce the notations $U := u_z$, $V := v_z$ and the complex function $W := U + \mathrm{i}V$, and we recast Eq. (26) as the single complex equation

$$(QW)_{zz} + \nu W_{zzt} + \alpha_1 W_{zztt} = \rho W_{tt}, \tag{28}$$

where Q is now a function of U^2+V^2 alone, $Q = Q(U^2+V^2)$. We decompose the complex function W into modulus and argument as

$$W(z,t) = \Omega(z,t)\exp(i\theta(x,t)), \tag{29}$$

say, and we seek solutions in the separable forms,

$$\Omega(z,t) = \Omega_1(z)\Omega_2(t), \quad \theta(x,t) = \theta_1(x) + \theta_2(t), \tag{30}$$

say. The solutions obtained from this ansatz are remarkable, first of all because they reduce the partial differential equation Eq. (28) to a system of *ordinary* differential equations, and also because they contain the generalization to a nonlinear setting of often encountered classes of wave solutions, such as damped and attenuated plane waves.

It is interesting to note that in some cases, the solutions of Eqs. (28) are similar to the classical viscoelastic case. For example, when we restrict our attention to the following generalized oscillatory shearing motion

$$W(z,t) = [\psi(t) + \mathrm{i}\phi(t)]\mathrm{e}^{\mathrm{i}(kz-\theta(t))}, \tag{31}$$

where k is a constant and ψ, ϕ, and θ are function of time alone, we recover for our present setting the same equations (and therefore the same solutions) as those already reported by Destrade and Saccomandi[8] in the context of compressible dissipative solids. Establishing a formal correspondence requires the replacement of their density ρ_0 with the positive quantity $\rho + k^2\alpha_1$ here; it also shows that no noticeable differences arise from the introduction of strong rate effects. For instance, such is the case when we consider, at $\nu = 0$, circularly polarized harmonic waves in the form

$$u(z,t) = A\cos(kz - \omega t), \quad v(z,t) = \pm A\sin(kz - \omega t). \tag{32}$$

Now, $U^2+V^2 = A^2k^2$ and therefore Q is independent of z; then the equation of motion Eq. (28) leads to the following dispersion equation:

$$Q(A^2k^2) = (\rho + k^2\alpha_1)\omega^2, \tag{33}$$

which is similar to the equation for the purely elastic case.

When we consider generalized shear sinusoidal standing waves[8], we obtain more noticeable results. Hence Carroll[9] took

$$u(z,t) = \phi(z)\cos\omega t, \quad v(z,t) = \phi(z)\sin\omega t, \tag{34}$$

where ϕ depends on z only. Then Eq. (28), written at $\nu = 0$, reduces to

$$\{[Q(\Phi^2) - \alpha_1\omega^2]\Phi\}'' + \rho\omega^2\Phi = 0, \tag{35}$$

where $\Phi := \phi'$. Here the main difference between this equation and the corresponding equation in the purely elastic case is the introduction of a *characteristic dispersive length* $\alpha_1\omega^2$. We also note that Eq. (35) is remarkable because the usual substitution for autonomous equations, $\Psi(\Phi) = \Phi'$, reduces this equation to a linear first order differential equation in Ψ^2, so that many solutions can be found in analytical form.

5. Concluding remarks

We showed in this note that dispersion and dissipation are strongly correlated for the solid of second grade. We provided several interesting exact solutions in special cases, and gave pointers on how to reduce the general dynamical equations down to ordinary differential equations. As it happens, some methods and results developed elsewhere in the literature can be applied to the present framework in a straightforward manner.

Our next step will be to extend the present model for a solid of second grade to a wider nonlinear setting, by taking the microstructural parameter α_1 to be a function of the invariants and no longer a constant. In such a way, it might be possible to balance dispersive and nonlinear effects, and perhaps to obtain some solitons and compactons. That derivation would constitute a major advancement in our understanding of the nonlinear mechanics of solids, with many important technical applications.

Acknowledgments

This work is supported by the French Ministère des Affaires Etrangères under the scheme: “Séjour Scientifique de Haut Niveau”.

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