FINITE-AMPLITUDE LOVE WAVES IN A PRE-STRESSED NEO-HOOKEAN MATERIAL

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Summary In the context of the non-linear elasticity theory we consider a model for compressible solids called "compressible neo-Hookean material". We show how (exact) finite-amplitude inhomogeneous plane wave solutions and finite-amplitude unattenuated solutions can combine to form a finite-amplitude Love wave. Also, we investigate the special case when the interface between the layer and the substrate is in a principal plane of the pre-strain.

COMPRESSIBLE NEO-HOOKEAN MATERIALS

In the context of the finite elasticity theory we consider a model for compressible solids called "compressible neo-Hookean material". These non-linear elastic materials are characterized by their mass density (in the undeformed state) ρ_0, by their shear modulus μ, and by a constitutive function $G(J)$ describing the compressibility properties. The constitutive equation for the Cauchy stress tensor $\mathbf{T}$ is

$$\mathbf{T} = \tfrac{1}{2}G'(J)\mathbf{1} + \mu J^{-1}\mathbf{B}\ , \qquad (1)$$

where $\mathbf{B}$ is the left Cauchy-Green strain tensor and $J = (\det \mathbf{B})^{1/2}$.

WAVE SUPERPOSED ON STATIC DEFORMATION

First we assume that a compressible neo-Hookean material is first subjected to a static finite homogeneous deformation $\mathbf{x} = \mathbf{F}\mathbf{X}$, where $\mathbf{F}$ is a constant deformation gradient. On this state of deformation we superpose a time-dependent displacement field representing a finite amplitude wave motion. Thus, a particle at position $\mathbf{x}$ moves to position $\bar{\mathbf{x}} = \mathbf{x} + \mathbf{u}(\mathbf{x}, t)$, where $\mathbf{u}$ is the mechanical displacement. We look for solutions with a displacement field of the form $\mathbf{u} = f(\mathbf{m}\cdot\mathbf{x})g(\mathbf{n}\cdot\mathbf{x} - vt)\mathbf{a}$, where f and g are functions to be determined, and $\mathbf{m}$, $\mathbf{n}$ and $\mathbf{a}$ are unit vectors. It is assumed that $\mathbf{m}$ and $\mathbf{n}$ are not parallel and that $\mathbf{a}$ is orthogonal to both $\mathbf{m}$ and $\mathbf{n}$, so that the displacement field $\mathbf{u}$ represents a linearly-polarized transverse wave with propagation speed v. It is shown that this displacement field is a solution of the equations of motion if and only if f and g satisfy the equation [1]

$$(\mathbf{n}\cdot\mathbf{B}\mathbf{n} - \mu^{-1}\rho_0 v^2)fg'' + 2\mathbf{n}\cdot\mathbf{B}\mathbf{m}f'g' + \mathbf{m}\cdot\mathbf{B}\mathbf{m}f''g = 0\ , \qquad (2)$$

where f' and g' denote the derivatives of f and g with respect to their argument. We then choose $\mathbf{m}$ and $\mathbf{n}$ such that [2] $\mathbf{n}\cdot\mathbf{B}\mathbf{m} = 0$, which means that $\mathbf{m}$ and $\mathbf{n}$ are along the principal axes ofhe elliptical section of the ellipsoid $\mathbf{x}\cdot\mathbf{B}\mathbf{x} = 1$ by the plane $\mathbf{a}\cdot\mathbf{x} = 0$. In such a case, we obtain two kinds of exact solutions, *unattenuated time-harmonic wave solutions*

$$\mathbf{u}(\mathbf{x}, t) = \left[B\sin(\frac{\kappa}{v_{\mathbf{m}}}\mathbf{m}\cdot\mathbf{x}) + C\cos(\frac{\kappa}{v_{\mathbf{m}}}\mathbf{m}\cdot\mathbf{x})\right]\cos\left(\frac{\kappa}{\sqrt{v^2 - v_{\mathbf{n}}^2}}(\mathbf{n}\cdot\mathbf{x} - vt)\right)\mathbf{a}\ , \qquad (3)$$

provided $v^2 > v_{\mathbf{n}}^2$, and *inhomogeneous time-harmonic wave solutions*

$$\mathbf{u}(\mathbf{x}, t) = A\exp\left(-\frac{\gamma}{v_{\mathbf{m}}}\mathbf{m}\cdot\mathbf{x}\right)\cos\left(\frac{\gamma}{\sqrt{v_{\mathbf{n}}^2 - v^2}}(\mathbf{n}\cdot\mathbf{x} - vt)\right)\mathbf{a}\ , \qquad (4)$$

provided $v^2 < v_{\mathbf{n}}^2$. Here, κ, B, C, γ, A are arbitrary constants, and $v_{\mathbf{m}}$, $v_{\mathbf{n}}$ are the wave speeds of homogeneous bulk waves propagating along $\mathbf{m}$ and $\mathbf{n}$, respectively, and are given by $\rho_0 v_{\mathbf{m}}^2 = \mu\,\mathbf{m}\cdot\mathbf{B}\mathbf{m}$, $\rho_0 v_{\mathbf{n}}^2 = \mu\,\mathbf{n}\cdot\mathbf{B}\mathbf{n}$.

FINITE-AMPLITUDE LOVE WAVES

Next we extend the classical results of Love [3] for linear elasticity theory in two directions: by taking into account initial stresses, and by allowing the wave amplitude to be arbitrarily large. We keep Love's original set-up, which consists of a semi-infinite substrate, covered with a layer of finite thickness h. The two solids are deformed and bonded rigidly. Here we assume that both the substrate and the layer are made of different 'compressible neo-Hookean materials', characterized by ρ_0, μ and G for the substrate, and by $\widetilde{\rho}_0$, $\widetilde{\mu}$ and $\widetilde{G}$ for the layer, and have been subjected to the homgeneous strains $\mathbf{B}$ and $\widetilde{\mathbf{B}}$, respectively. It is seen that $\mathbf{B}$ in the substrate is determined from $\widetilde{\mathbf{B}}$ in the layer, and here we assume that $\mathbf{n}\cdot\mathbf{B}\mathbf{m} = \mathbf{n}\cdot\widetilde{\mathbf{B}}\mathbf{m} = 0$. In particular, it follows $\mathbf{n}\cdot\mathbf{B}\mathbf{n} = \mathbf{n}\cdot\widetilde{\mathbf{B}}\mathbf{n}$.

As in the classical linear case, we focus on linearly-polarized transverse waves, propagating in a direction $\mathbf{n}$ and polarized in a transverse direction $\mathbf{a}$, both parallel to the interface. Thus, $(\mathbf{n}, \mathbf{a}, \mathbf{m})$ forms an orthonormal triad. In the substrate, we require that the amplitude of the wave decays in the direction of $\mathbf{m}$, and hence the displacement field $\mathbf{u}$ is assumed to be of the form (4). In the layer, we consider an unattenuated time-harmonic displacement field $\widetilde{\mathbf{u}}$ of the form (3). As in the classical case, we combine these exact wave solutions in order to obtain a global time-harmonic wave motion with propagation speed v. Note that both displacement fields need to be of the same angular frequency, and hence of the same wavenumber k, in order to satisfy boundary conditions. Thus, we use (4) for the substrate and (3) for the layer, with

$$k = \frac{\kappa}{\sqrt{v^2 - \widetilde{v}_{\mathbf{n}}^2}} = \frac{\gamma}{\sqrt{v_{\mathbf{n}}^2 - v^2}}\ , \tag{5}$$

where the body waves speeds $v_{\mathbf{n}}$, $v_{\mathbf{m}}$, $\widetilde{v}_{\mathbf{n}}$, $\widetilde{v}_{\mathbf{m}}$ are given by $\rho_0 v_{\mathbf{m}}^2 = \mu \mathbf{m} \cdot \mathbf{Bm}$, $\rho_0 v_{\mathbf{n}}^2 = \mu \mathbf{n} \cdot \mathbf{Bn}$, $\widetilde{\rho}_0 \widetilde{v}_{\mathbf{m}}^2 = \widetilde{\mu} \mathbf{m} \cdot \widetilde{\mathbf{B}}\mathbf{m}$, $\widetilde{\rho}_0 \widetilde{v}_{\mathbf{n}}^2 = \widetilde{\mu} \mathbf{n} \cdot \mathbf{Bn}$. The Love wave speed v has to satisfy $\widetilde{v}_{\mathbf{n}}^2 < v^2 < v_{\mathbf{n}}^2$; thus, Love waves require the combination of a 'slow' (or 'soft') layer over a 'fast' (or 'hard') substrate, independently of the initial pre-strain. We show that the expression of the boundary conditions lead to a dispersion equation and to the determination of the constants A, B, C in terms of a single parameter characterizing the amplitude of the wave. The first and second boundary conditions require the continuity of the displacement and of the stress vector at the layer/substrate interface, and the third condition requires that there is no additional traction at the upper face of the layer. These conditions lead to the dispersion equation

$$\tan\left[kh(\widetilde{v}_{\mathbf{n}}/\widetilde{v}_{\mathbf{m}})\sqrt{(v/\widetilde{v}_{\mathbf{n}})^2 - 1}\right] - \frac{\rho_0 c}{\widetilde{\rho}_0 \widetilde{c}} \sqrt{\frac{(c/\widetilde{c})^2 - (v/\widetilde{v}_{\mathbf{n}})^2}{(v/\widetilde{v}_{\mathbf{n}})^2 - 1}} = 0\ , \tag{6}$$

where c and $\widetilde{c}$ are the transverse bulk wave speeds in the undeformed substrate and layer, respectively. This dispersion equation is similar to that of the linear case, but the scope of the results is now richer because they include large amplitude and pre-stress.

INTERFACE IN A PRINCIPAL PLANE

For a given static strain $\widetilde{\mathbf{B}}$ in the layer and a given unit vector $\mathbf{m}$, there is, in general, only one direction $\mathbf{n}$ in the interface along which a finite-amplitude Love wave as described previously may propagate. However, if $\mathbf{m}$ is along a principal axis of $\widetilde{\mathbf{B}}$, then $\mathbf{n} \cdot \mathbf{Bm} = \mathbf{n} \cdot \widetilde{\mathbf{B}}\mathbf{m} = 0$ is satisfied for any propagation direction $\mathbf{n}$ orthogonal to $\mathbf{m}$ and so, $\mathbf{n}$ may be along any direction in the interface. It is shown that the number of possible modes for a given value of the dispersion parameter kh is not necessarily the same for all $\mathbf{n}$. In the figure, a value of kh has been chosen so that two modes with speeds v_1, v_2 are possible for all directions $\mathbf{n}$.

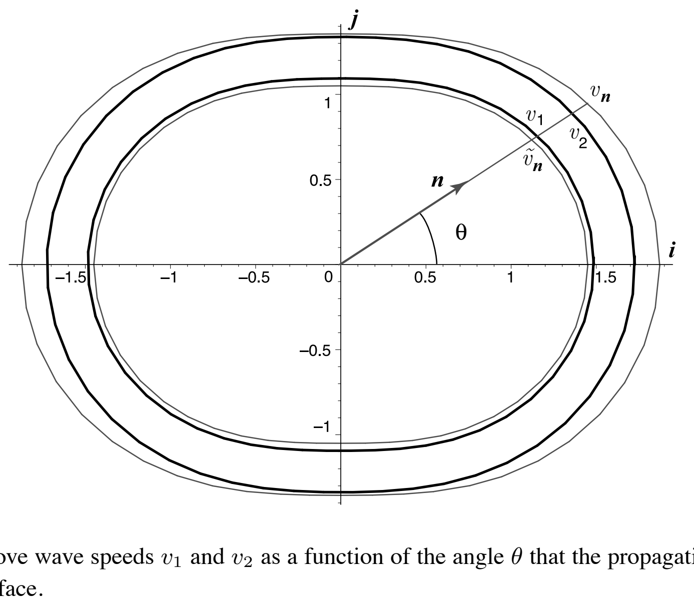

Figure 1. Polar graph of the Love wave speeds v_1 and v_2 as a function of the angle θ that the propagation direction $\mathbf{n}$ makes with the principal direction $\mathbf{i}$ in the interface.

References

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