BACHET'S PROBLEM: AS FEW WEIGHTS TO WEIGH THEM ALL

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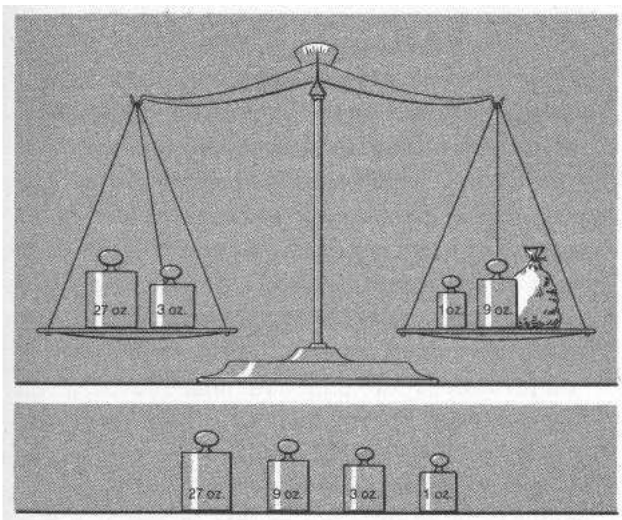
The genesis of many areas in mathematics can often be found in some simply-put puzzle, a word problem that doesn't require any formal language or concise definitions to understand. A few cases in point are graph theory having its origins in Euler's *Bridges of Königsberg problem*, the *Chinese remainder problem* which best captures the rules of modular arithmetic in number theory & abstract algebra, and the dark arts of probability having their roots in 17^{th}-century games of chance. Generalizations of these problems form the bedrock for much of what came afterward.

In other subjects, progress is made instead with the root problems leading to others but without those root problems ever being solved. This is the case in number theory where the resolution of some its first problems like Goldbach's conjecture and the twin primes conjecture remain unsolved, mostly because they are too hard. But it can also be the case that the first problem of a modern and active area of mathematics can simply be forgotten as that, even if the problem enjoys an enduring popularity both within and outside the mathematics classroom. This is certainly the case with the problem that we will generalize here and which we argue should be regarded as one of the first problems, if not the first, of the thoroughly modern area of integer partitions:

What is the least number of pound weights that can be used on a scale pan to weigh any integral number of pounds from 1 to 40 inclusive, if the weights can be placed in either of the scale pans ?

W.W. Rouse Ball [4, pp.50] attributes the first recording of this problem to Bachet in the early 17th century, calling it *Bachet's Weights Problem*, and Hardy & Wright thought it fit to include it in their wonderful and highly influential *An introduction to the theory of numbers* [11]. However, Bachet's problem stretches all the way back to Fibonacci [17, *On IIII Weights Weighing Forty Pounds]* in 1202!



1 = 1	21 = 27 − 9 + 3
2 = 3 − 1	22 = 27 − 9 + 3 + 1
3 = 3	23 = 27 − 3 − 1
4 = 3 + 1	24 = 27 − 3
5 = 9 − 3 − 1	25 = 27 − 3 + 1
6 = 9 − 3	26 = 27 − 1
7 = 9 − 3 + 1	27 = 27
8 = 9 − 1	28 = 27 + 1
9 = 9	29 = 27 + 3 − 1
10 = 9 + 1	30 = 27 + 3
11 = 9 + 3 − 1	31 = 27 + 3 + 1
12 = 9 + 3	32 = 27 + 9 − 3 − 1
13 = 9 + 3 + 1	33 = 27 + 9 − 3
14 = 27 − 9 − 3 − 1	34 = 27 + 9 + 1 − 3
15 = 27 − 9 − 3	35 = 27 + 9 − 1
16 = 27 − 9 − 3 + 1	36 = 27 + 9
17 = 27 − 9 − 1	37 = 27 + 9 + 1
18 = 27 − 9	38 = 27 + 9 + 3 − 1
19 = 27 − 9 + 1	39 = 27 + 9 + 3
20 = 27 − 9 + 3 − 1	40 = 27 + 9 + 3 + 1

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Bachet's problem needs no more than four weights and these (unique) pound weights are 1,3,9 and 27. The figure [22, pp. 53] displays how to weigh 20 (Steinhaus [22] had the good sense to only lift ounce rather than pound weights onto the page) and the table (also from [22, pp. 53]) displays how to measure all the weights between 1 and 40 inclusive, a positive coefficient assigned to weights placed on the left scale, a negative to those on the right. Writing the solution as an integer partition with four parts $40 = 1 + 3 + 9 + 27$, Bachet's problem's noble roots in *Fibonacci's Liber Abaci* [17] make it a viable candidate for the first problem of integer partitions.

The *generalized Bachet's problem* that we will explore here is that of finding appropriate weights when one replaces 40 with any positive integer. Until relatively recently the only generalizations known were that of replacing 40 with integers of the form $\frac{1}{2}(3^{n+1} - 1)$ [11, §9.7]. The full generalization, due to Park [16] and studied further by Rødseth [19], not only tells us the *minimum number of parts* needed when 40 is replaced by any m but *all possible ways to accordingly break up* a given m. Furthermore, how we can *count* the number of distinct ways to break up such an m. For example, when we replace 40 by $m = 25$ we'll still need no more than four parts but there are now nine ways to break up 25 to solve Bachet's problem. Written as partitions with four parts, these are:

$$\begin{array}{lllllll} 25 & = & 1+3+9+12 & = & 1+3+8+13 & = & 1+3+7+14 \\ & = & 1+3+6+15 & = & 1+3+5+16 & = & 1+3+4+17 \\ & = & 1+2+7+15 & = & 1+2+6+16 & = & 1+2+5+17 \end{array}$$

Remarkably, given the age and popularity of Bachet's problem, these headways have come to light only in the last fifteen or so years and they seem to be little known at that. Given its status as one of the first problems of partitions of integers, the present piece aims to rectify this sad state of affairs and to do so in a lively and informal yet unambiguous fashion.

We will also expound on similar problems like generalizing the following: *what is the least number of pound weights that can be used on a scale pan to weigh any integral number of pounds from 1 to 15 inclusive, if the weights can be placed in only one of the scale pans ?* This one-scale problem was my first encounter with Bachet's problem and what started as a simple recreational problem (first heard from a carpenter who had heard it on a construction site) rapidly turned into an introduction to the wonders of partitions of integers, recurrence relations, generating functions and counting integer points in polyhedra. My hope here is that we can repeat that same journey, using only our sharp wits and a willingness to induct!

Finally, and fittingly considering his substantial and eclectic contribution to both recreational mathematics and modern combinatorics, we will close with MacMahon's generalization of (the two-scale) Bachet's problem: he noticed [12] that $1, 3, 9, 27$ can be used to *uniquely* weigh every integer weight between 1 and 40. For example, the figure displays that $20 = -1 + 3 - 9 + 27$ and we claim, in the sense of Bachet, that this is the only way to write 20 using $1, 3, 9$ and 27. We will see what the factorization $3 \times 3 \times 3 \times 3$ of 81 has to do with the weight set $1, 3, 9, 27$ for 40, and much more.

A first solution to the generalized Bachet's problem

Before becoming a touch more formal, let's provide a taster of what's to come by providing our first candidates, one candidate of mostly ternary weights for each positive integer m, to solve the generalized Bachet's problem. Given a positive integer m there is a unique integer n such that $\frac{1}{2}(3^n - 1) + 1 \leq m \leq \frac{1}{2}(3^{n+1} - 1)$. We can break the integer m into $n + 1$ smaller integer weights consisting of those elements in the multi-set $\mathcal{W}_m := \{1, 3, 3^2, \ldots, 3^{n-1}, m - (1 + 3 + 3^2 + \cdots + 3^{n-1})\}$.

For example, if $m = 25$ then

$$14 = \frac{1}{2}(3^3 - 1) + 1 \leq 25 \leq \frac{1}{2}(3^{3+1} - 1) = 40 \text{ and } \mathcal{W}_{25} = \{1, 3, 9, 25 - (1 + 3 + 9)\} = \{1, 3, 9, 12\}.$$

Proposition 1. *Every integer weight l with $0 \leq l \leq m$ can be measured using a two scale balance with the weights from the multiset $\mathcal{W}_m$.*

To see that this is true in the case of $m = 25$ observe that every integer in the closed interval $[-13, 13]$ can be measured using $\{1, 3, 9\}$. With the extra weight of 12 we can in addition measure every integer in the shifted closed interval $12 + [-13, 13] = [-1, 25]$ and so the proposition holds for $m = 25$. Let's prove it now for every m.

Proof. For $m = 1, 2, 3, 4$ (those m's with $n = 0$ or 1) we have $\mathcal{W}_1 = \{1\}$, $\mathcal{W}_2 = \{1, 1\}$, $\mathcal{W}_3 = \{1, 2\}$ and $\mathcal{W}_4 = \{1, 3\}$ respectively and, for every such m, every $0 \leq l \leq m$ can be measured using both pans of the two scale balance with the weights in $\mathcal{W}_m$. Assume that this is the case for every $m \leq \frac{1}{2}(3^n - 1)$. In particular, assume that $\mathcal{W}_{\frac{1}{2}(3^n-1)} := \{1, 3, 3^2, \ldots, 3^{n-1}\}$ can be used to weigh every integer l in the closed interval $[-\frac{1}{2}(3^n - 1), \frac{1}{2}(3^n - 1)]$ – thinking in terms of the two scale pans, a negative $-l$ would have the weights on the scales interchanged from that of the positive l.

We will now proceed by induction on n to show that every $l \leq m$ can be measured (using both pans of the two scale balance) with the weights in $\mathcal{W}_m$ for all m's with $\frac{1}{2}(3^n - 1) + 1 \leq m \leq \frac{1}{2}(3^{n+1} - 1)$. Since the multiset $\mathcal{W}_{\frac{1}{2}(3^n-1)}$ is contained in $\mathcal{W}_m$ then, by our inductive hypothesis, every integer in the closed interval $[-\frac{1}{2}(3^n - 1), \frac{1}{2}(3^n - 1)]$ can be measured by using weights from $\mathcal{W}_m \backslash \{m - \frac{1}{2}(3^n - 1)\}$ on the two scale balance. Consequently, every integer weight in the following closed interval can be measured using $\mathcal{W}_m$:

$$m - \frac{1}{2}(3^n - 1) + [-\frac{1}{2}(3^n - 1), \frac{1}{2}(3^n - 1)] = [m - 3^n + 1, m].$$

When combined with our induction hypothesis, this implies that all integers in the union of the closed intervals $[0, \frac{1}{2}(3^n - 1)] \cup [m - 3^n + 1, m]$ can be measured using $\mathcal{W}_m$. Now recall that $m \leq \frac{1}{2}(3^{n+1} - 1)$ which implies that

$$m - 3^n + 1 \leq \frac{1}{2}(3^{n+1} - 1) - 3^n + 1 = \frac{1}{2}(3^n - 1) + 1$$

and so the integers in the set $[0, \frac{1}{2}(3^n - 1)] \cup [m - 3^n + 1, m]$ are precisely those integers in the set $[0, m]$. In other words, every integer weight l with $0 \leq l \leq m$ can be measured using a two scale balance with the weights from $\mathcal{W}_m$. □

In the case of $m = \frac{1}{2}(3^{n+1} - 1)$, the above proposition was intimated by Fibonacci in [17, *On IIII Weights Weighing Forty Pounds*] and first proved by Hardy & Wright [11, §9.7] who went further by showing that $\mathcal{W}_{\frac{1}{2}(3^{n+1}-1)}$ is not only the smallest multiset of weights that satisfy the Bachet problem for $m = \frac{1}{2}(3^{n+1} - 1)$ but that it is the unique such multiset.

In the next sections, we will see that $\mathcal{W}_m$ is a multiset of *minimal* size with the property that every weight between 0 and m can be measured using a two scale balance and from this analysis Hardy & Wright's claim of $\mathcal{W}_{\frac{1}{2}(3^{n+1}-1)}$ being the unique such multiset will follow. But in order to do so we will need first to delve into the language of partitions of integers.

PARTITIONS OF INTEGERS

Luckily for us, the description of all solutions to the generalized Bachet problem is surprisingly elegant and simple when phrased in terms of *partitions of integers*. Let's survey the tip of the iceberg that is integer partitions. A *partition* of a positive integer m is an ordered sequence of positive integers that sum to m: $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ with $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. We call the $n+1$ λ_i's the *parts* of the above partition. For example, 5 has seven distinct partitions given by

$$5 = 1+1+1+1+1 = 1+1+1+2 = 1+2+2 = 1+1+3 = 1+4 = 2+3$$

and we denote this by $p(5) = 7$. Analogous to the hand-shaking lemma in graph theory, the first lemma that everyone encounters in integer partitions is: the number of partitions of a given m with no parts larger than $n+1$ equals the number of partitions of m with at most $n+1$ parts. For $m = 5$ and $n+1 = 2$ this translates to $|\{1+1+1+1+1,\ 1+1+1+2,\ 1+2+2\}| = |\{5,\ 1+4,\ 2+3\}|$. See [3] for a first introduction to integer partitions and [2] for a more advanced perspective.

Returning to Bachet's problem, let's call a partition of m a *Bachet partition* if

(1) every integer $0 \leq l \leq m$ can be written as $l = \sum_{i=0}^{n} \beta_i \lambda_i$ where each $\beta_i \in \{-1, 0, 1\}$.

and (2) there does not exist another partition of m satisfying (1) with fewer parts than $n+1$.

For example, only four of the seven partitions of 5 satisfy condition (1): $\{1+1+1+1+1,\ 1+1+1+2,\ 1+2+2,\ 1+1+3\}$. And of these four partitions only two have the fewest possible number of three parts: $\{1+2+2,\ 1+1+3\}$. In short, 5 has two Bachet partitions.

Another example is the partition $1+3+9+12$ of 25, whose parts are precisely the elements of $\mathcal{W}_{25}$. Proposition 1 above amounts to saying that this partition satisfies condition (1). What remains to be shown is whether this partition satisfies (2). We could of course list all $p(25) = 1958$ partitions of 25 [20, A000041] and check which of those satisfy (1). And then pick out those with the fewest number of parts just as we did above for finding the Bachet partitions of 5. But with some simple observations about condition (1) above we'll soon be able to do much better than this brute-force, tedious computation.

Noting that condition (1) above involves positive and negative β_i coefficients it can be beneficial to only have to worry about addition and to do so we can rewrite condition (1) as:

$(1)'$ every integer $0 \leq l \leq 2m$ can be written as $l = \sum_{i=0}^{n} \alpha_i \lambda_i$ where each $\alpha_i \in \{0, 1, 2\}$.

The equivalence of conditions (1) & $(1)'$, as essentially noted by Hardy & Wright in [11, §9.7], is given by the shift of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_m$ in $l - m = \sum_{i=0}^{n} \alpha_i \lambda_i - \sum_{i=0}^{n} \lambda_i = \sum_{i=0}^{n} \beta_i \lambda_i$. Note that we could just as easily have replaced $0 \leq l \leq m$ in (1) with $-m \leq l \leq m$ since, thinking in terms of the two scales, a negative $-l$ would have the weights on the scales interchanged from that of the positive l.

Partitions of an integer m satisfying $(1)'$ are called 2-*complete partitions* and were introduced by Park [16] only twelve years ago. This shift between conditions (1) and $(1)'$ is little more than a sleight of hand but it does resolve the central difficulty in dealing with (1), in that it avoids having to deal with both addition & subtraction operations, whereas $(1)'$ involves only addition. We'll see in the next section that condition $(1)'$ immediately tells us that $\lambda_0 = 1$ but this is not as obvious when using only (1). Much more will also become transparent from this formulation in the next section where we resolve the minimality of parts condition.

THE MINIMALITY OF PARTS CONDITION

A simple equivalence regarding the 2-complete partitions, first proved by Park, will amazingly tell us all that we need to know about Bachet partitions. We'll deal first with the minimality condition (2).

Lemma 2. [16] *If* $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ *is a 2-complete partition then* $\lambda_0 = 1$ *and* $\lambda_i \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ *for every* $i = 1, 2, \ldots, n$.

Proof. Since $0 \leq 1 \leq 2m$ then we must be able to write 1 as a $\{0, 1, 2\}$-combination of the parts $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ and if $\lambda_0 \geq 2$ then such a $\{0, 1, 2\}$-combination of the parts would be impossible. Hence, $\lambda_0 = 1$ as claimed.

Consider next, for each $i = 1, \ldots, n$, the non-negative integer $\lambda_i - 1$. Since $\lambda_i - 1 < \lambda_i \leq \ldots \leq \lambda_n$ and since $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is a 2-complete partition then there must exist a $\{0, 1, 2\}$-combination of the parts $\lambda_0, \lambda_1, \ldots, \lambda_{i-1}$ that equals $\lambda_i - 1$. Hence $\lambda_i - 1$ cannot exceed the largest of all $\{0, 1, 2\}$-combinations of $\lambda_0, \lambda_1, \ldots, \lambda_{i-1}$, which would be $2\lambda_0 + 2\lambda_1 + \cdots + 2\lambda_{i-1}$. In other words, $\lambda_i \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ as claimed. □

Corollary 3. *If* $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ *is a 2-complete partition then* $\lambda_i \leq 3^i$ *for every* $i = 0, 1, \ldots, n$.

This corollary follows by first noting that if $\lambda_0 = 1$ then $\lambda_1 \leq 1 + 2(1) = 3$. In turn, $\lambda_2 \leq 1 + 2(1 + 3) = 9$ and the corollary now follows by an inductive argument. Now we come to the minimality condition of Bachet partitions. Corollary 3 implies that if $m = \lambda_0 + \lambda_1 + \cdots \lambda_n$ is a Bachet partition then the sum of the parts in the partition cannot exceed $\sum_{i=0}^{n} 3^i = \frac{1}{2}(3^{n+1} - 1)$. That is,

$$m \leq \frac{1}{2}(3^{n+1} - 1) < \frac{1}{2}3^{n+1} \quad \text{or} \quad \log_3(2m) < n + 1.$$

Since $n + 1$ is an integer then the integer part of $\log_3(2m)$ i.e. $\lfloor \log_3(2m) \rfloor < n + 1$ (the function $\lfloor x \rfloor$ takes a real number x to the greatest integer that is less than or equal to x). Since both $\lfloor \log_3(2m) \rfloor$ and $n + 1$ are integers then $\lfloor \log_3(2m) \rfloor \leq n$. In summary, Corollary 3 tells us that a Bachet partition must have *at least* $\lfloor \log_3(2m) \rfloor + 1$ parts. So if we could find a partition satisfying condition (1) with exactly $\lfloor \log_3(2m) \rfloor + 1$ parts then $\lfloor \log_3(2m) \rfloor + 1$ must be precisely the number of parts needed for a Bachet partition of m.

But we do have such a partition! The elements of the multiset $\mathcal{W}_m$ from Proposition 1 , reordered in increasing order and set equal (in order) to λ_0 through λ_n, make such a partition. For example, $25 = 1 + 3 + 9 + 12$ is a Bachet partition because of Proposition 1 combined with Corollary 3.

Theorem 4. *A Bachet partition of a positive integer* m *has precisely* $\lfloor \log_3(2m) \rfloor + 1$ *parts.*

This theorem was essentially stated in [16] and formally stated, including Proposition 1, by Rødseth [19, Lemma 3.2] where Bachet partitions are called *minimal 2-complete partitions*. One might wonder next: are the Bachet partitions from Proposition 1 the only Bachet partitions for each positive integer m ? In the case of $m = \frac{1}{2}(3^{n+1} - 1)$ it is now easy to show that the answer is yes. To see this let $\frac{1}{2}(3^{n+1} - 1) = \lambda_0 + \lambda_1 + \cdots \lambda_n$ be a Bachet partition. If any of the λ_j's were strictly less than 3^j then from Corollary 3 we would have $\frac{1}{2}(3^{n+1} - 1) = \sum_{i=0}^{n} \lambda_i < \frac{1}{2}(3^{n+1} - 1)$ which cannot occur. Hence, as claimed by [11, §9.7], $1 + 3 + 3^2 + \cdots + 3^n$ is the unique Bachet partition for $m = \frac{1}{2}(3^{n+1} - 1)$.

In the next section, we will show that a partition is a Bachet partition if and only if it both has the number of parts as stated above and, amazingly, the conclusion of Lemma 2 is satisfied for all parts in the partition. But before doing so permit us to digress a little and say what was so enjoyable about this section: we discovered everything we needed to know about the number of parts needed for a Bachet partition by starting with a very simple collection of inequalities (Lemma 2) and then we used a very generous version of these inequalities to attain $\lambda_i \leq 3^i$. When combined with Proposition 1 we were able to solve the problem of the number of weights needed for the Bachet problem.

While not containing content that is anywhere near as substantial, this "generous use" of inequalities reminds us of one of the reasons why Euler's proof that there are infinitely many primes [1, Ch.1, 4^{th} proof] is many a person's favorite: Euler seems to throw out everything but the kitchen sink with every relaxed inequality made but he still ends up capturing a very sharp picture. With such languid inequalities, he shows that the number of primes less than a real number x is bounded below by $\log_e(x) - 1$ which was the first prophecy of the prime number theorem on how primes are distributed asymptotically. We should know better but it is still surprising to attain meaningful, sharp results from relaxed inequalities such as those used in this section. See [21] for a delightful, analysis-flavored account on all things being "inequal"!

Bachet partitions as lattice points in polyhedra

Recall our example of $25 = 1 + 3 + 9 + 12$ as a Bachet partition. In contrast to the scenario where $\frac{1}{2}(3^{n+1} - 1) = 1 + 3 + 3^2 + \cdots + 3^n$ is a unique Bachet partition for that particular m, there are many (many being nine!) Bachet partitions for $m = 25$:

$$\begin{array}{rclclcl} 25 & = & 1+3+9+12 & = & 1+3+8+13 & = & 1+3+7+14 \\ & = & 1+3+6+15 & = & 1+3+5+16 & = & 1+3+4+17 \\ & = & 1+2+7+15 & = & 1+2+6+16 & = & 1+2+5+17 \end{array}$$

That these partitions are precisely the Bachet partitions for 25 follow from this remarkable result:

Theorem 5. (Park [16, Theorem 2.2]) *The partition $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is a Bachet partition if and only if $n = \lfloor \log_3(2m) \rfloor$, $\lambda_0 = 1$ and $\lambda_i \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for every $i = 1, 2, \ldots, n$.*

Proof. Due to Lemma 2 and Theorem 4 all we need show is that if $\lambda_i \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for every $i = 1, 2, \ldots, n$ then $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is a 2-complete partition. This will be carried out by induction on the number of parts in the partition. Let $\mathcal{S}_n$ be the set of all partitions with $n+1$ parts that satisfy $\lambda_0 = 1$ and $\lambda_i \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for every $i = 1, 2, \ldots, n$.

We will show that $\mathcal{S}_n$ is contained in the set of 2-complete partitions. Clearly this is true for $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_1 = \{1+1, 1+2, 1+3\}$. Assume it is so for all $\mathcal{S}_i$'s where $i \leq n-1$. We will show that $\mathcal{S}_n$ is contained in the set of 2-complete partitions. Let $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ be a fixed partition in $\mathcal{S}_n$. Note that this implies that $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1}$ is in $\mathcal{S}_{n-1}$ and so our inductive hypothesis tells us the following: every integer less than or equal to $2(\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_{n-1})$ can be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$.

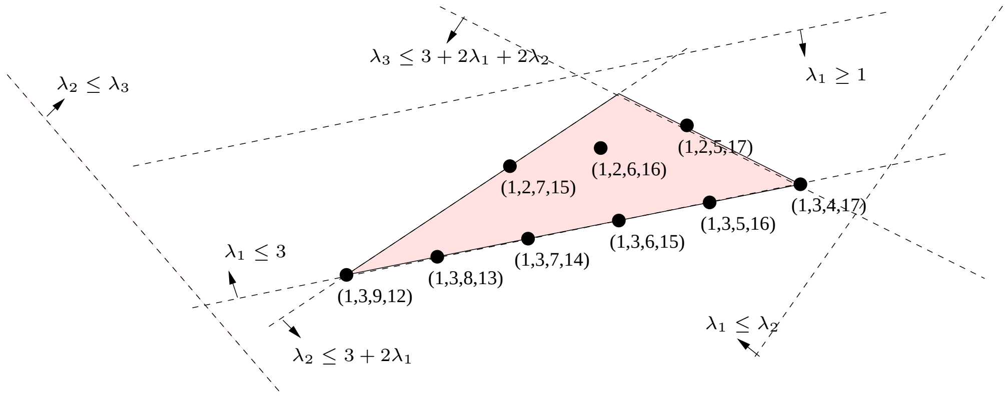
Let l be any positive integer no larger than $2(\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n)$. We wish to show that l can be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_n$. If l equals λ_n or $2\lambda_n$ then there is nothing to show. Otherwise either $l < \lambda_n$, or $\lambda_n < l < 2\lambda_n$, or $l > 2\lambda_n$.

If $l < \lambda_n$ then since $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is in $\mathcal{S}_n$ we have $l < \lambda_n \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1})$ which implies that $l \leq 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1})$ and from our inductive hypothesis as observed above l can then be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$.

If $\lambda_n < l < 2\lambda_n$ then $0 < l - \lambda_n < \lambda_n$ and $\lambda_n \leq 1 + 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1})$. Just as in the $l < \lambda_n$ case, our inductive hypothesis guarantees that $l - \lambda_n$ can then be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ and so l can then be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$.

Finally, it is assumed that $l \leq 2(\lambda_0 + \lambda_1 + \cdots + \lambda_n)$ and so if $2\lambda_n < l$ we then have $0 < l - 2\lambda_n \leq 2(\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1})$ and so, just as in the cases above, l can then be written as a $\{0, 1, 2\}$-combination of $\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{n-1}, \lambda_n$. □

One striking aspect of this inequality formulation of the Bachet partitions for m is that for each positive m we can think of the Bachet partitions of a given m as the set of lattice points (points all of whose entries are integers) in the polyhedron in $\mathbb{R}^n$ defined by the inequalities of Theorem 5. The nine Bachet partitions of 25 written as $(\lambda_0, \lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^4$ sit in the (two-dimensional) plane living in $\mathbb{R}^4$ cut out by the equations $\lambda_0 = 1$ and $\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 25$ and by the six additional *halfspaces* defined by the six inequalities: $\lambda_i \leq 1 + 2(\lambda_0 + \cdots + \lambda_{i-1})$ for each $i = 1, 2, 3$ and $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3$. By *cut out* we really mean the region in $\mathbb{R}^4$ given by the intersection of the two three-dimensional planes and the six halfspaces:



The first three inequalities (along with the two-dimensional plane) cut out the shaded triangle shown. In this case of $m = 25$, the other three inequalities that define the ordering of the parts of the partitions are not needed – they are said to be *redundant* – as they do not contribute to the *cutting out* of the shaded triangle. The Bachet partitions of 25, as expected from Theorem 5, are precisely the integer points in the shaded region.

Another striking consequence of the inequality formulation is that every Bachet partition has an *hereditary property*: it can be both projected down to and lifted up from another Bachet partition. In the next sections we'll use this hereditary property to count the number of Bachet partitions for a given m but in order to do so we must first talk about ternary partitions and generating functions.

PRECURSOR TO COUNTING: TERNARY PARTITIONS AND GENERATING FUNCTIONS

An important general problem in mathematics is that of counting the number of lattice points in a given polyhedron. The textbook of Beck & Robins [6] provides a wonderful, accessible introduction to this problem and how it arises in many contexts like discrete geometry, number theory and combinatorics. As seen in the previous section, counting the number of Bachet partitions for a given m is precisely this problem where the polyhedron is defined by the constraints described in Theorem 5. There is a formula due to Rødseth [19, Theorem 2.1] for counting precisely the number of distinct Bachet partitions for a given m. It is a *generating function* formula and a quick perusal of the encyclopedic [2] should convince the reader that generating functions are *the* standard way of counting in the theory of partitions of integers.

However, the full derivation [19, §4] of Rødseth's formula is difficult and technical and is beyond the scope (and against the informal spirit) of this present article. But we wil describe Rødseth's formula nonetheless and justify it for a substantial number of cases. To describe it we'll first need to talk about *ternary partitions* and their generating function.

Recall that we can formally write the geometric series $1 + x^t + x^{2t} + x^{3t} + x^{4t} + \cdots$ as $\frac{1}{1-x^t}$. We define the generating function

$$F(x) := \sum_{k=0}^{\infty} f(k)x^k = \prod_{i=0}^{\infty} \frac{1}{1-x^{3^i}}.$$

where $f(k)$ is understood as the coefficient of x^k in the infinite product $(1 + x + x^2 + x^3 + x^4 + \cdots)(1 + x^3 + x^6 + x^9 + x^{12} + \cdots)(1 + x^9 + x^{18} + x^{27} + x^{36} + \cdots)\cdots$. The significance of the term *generating function* comes from each $f(k)$ counting the k^{th} instance of some combinatorial phenomenon; in this case, the number of partitions of k into powers of 3 (these are called *ternary partitions*). For example, $f(15) = 9$ since there are precisely nine partitions of 15 all of whose parts are powers of 3:

$$\overbrace{1+\cdots+1}^{15\text{ times}},\ \overbrace{1+\cdots+1}^{12\text{ times}}+3,\ \overbrace{1+\cdots+1}^{9\text{ times}}+3+3,\ 1+1+1+1+1+1+3+3+3,$$
$$1+1+1+3+3+3+3,\ 1+1+1+1+1+1+9,\ 1+1+1+3+9,\ 3+3+3+3+3,\ 3+3+9.$$

A contribution of "1" is made to the coefficient $f(15) = 9$ for each ternary partition of 15. One contribution of 1 would be given by the term $x^3x^{4(3)}$ which represents the ternary partition $15 = 1 + 1 + 1 + 3 + 3 + 3 + 3$ of three 1's and four 3's. By convention, $f(0) = 1$.

The generating function $F(x)$ also satisfies the functional equation $F(x) = \frac{1}{(1-x)}F(x^3)$ or, $(1 - x)F(x) = F(x^3)$ and looking at the coefficient of x^{3k} in this equation we attain a recurrence relation $f(3k) - f(3k - 1) = f(k)$ or

$$f(3k) = f(3k - 1) + f(k).$$

Returning to our ternary partitions this recurrence should not be so surprising: it says that the ternary partitions of $3k$ can be made from those of $3k - 1$ (all of these already contain at least two 1's as parts and so adding another part equal to 1 gives all possible ternary partitions of $3k$ with some parts equal to 1) and from those ternary partitions of k (by multiplying all terms of these ternary partitions of k by 3 we get ternary partions of $3k$ with no parts equal to 1). The recurrence relation $f(3k) = f(3k - 1) + f(k)$ explains this manner of counting the ternary partitions of $3k$ in a concise and unfussy manner.

In other words, the generating function $F(x)$ is not only an accounting mechanism for ternary partitions but we can also manipulate the properties of $F(x)$ to recover encoded information about the ternary partitions themselves. These are some of the reasons that generating function formulae are thought of as the most useful means of counting not only specific partitions of integers but other combinatorial phenomena. A wonderfully colorful yet precise introduction to generating functions in general (and much, much more) is [10] and [3] introduces them in the context of partitions of integers.

Returning again to our recurrence relations for the ternary partitions, we can observe that that $f(3k) = f(3k+1) = f(3k+2)$ since the ternary partitions of $3k+1$ and $3k+2$ are those given by adding one and two extra parts equal to 1 respectively to those of $3k$. We can thus generalize the recurrence relation $f(3k) = f(3k-1) + f(k)$ to $f(k) = f(k-3) + f(\lfloor \frac{k}{3} \rfloor)$. By the same recurrence we have $f(k-3) = f(k-6) + f(\lfloor \frac{k-3}{3} \rfloor) = f(k-6) + f(\lfloor \frac{k}{3} \rfloor - 1)$ and repeating we have

$$f(k) = \sum_{i=0}^{\lfloor \frac{k}{3} \rfloor} f(i)$$

with the initial condition of $f(0) = 1$. As we would expect, this recurrence yields $f(2) = f(1) = f(0) = 1$ and so $f(5) = f(4) = f(3) = f(1) + f(0) = 2$ which yields $f(15) = 1+1+1+2+2+2 = 9$ as claimed from the generating function above.

COUNTING BACHET PARTITIONS: PROJECTING DOWN AND LIFTING UP

Let us now explain the link between Bachet partitions and ternary partitions. Letting Bachet(m) denote the set of Bachet partitions of m, Rødseth's formula amounts to showing that

$$|\text{Bachet}(m)| = f(\frac{1}{2}(3^{n+1} - 1) - m)$$

for essentially two-thirds of all positive integers m. For the other one-third, we will also describe what happens, in terms of counting lattice points in polyhedra.

We begin with two observations. The first is that we get Bachet partitions from $1+2+7+15$ by sequentially peeling off (projecting down) their largest parts: $1+2+7+15 \longrightarrow 1+2+7 \longrightarrow 1+2 \longrightarrow 1$. It's clear in this example that every peeling will project to a unique Bachet partition and this *hereditary property* is true in general: if $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is a Bachet partition then so is $\lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_j$ for every $j = 0, 1, \ldots, n-1$. The only tedious part of the proof of this claim is showing that $\frac{1}{2}(3^j - 1) + 1 \le \lambda_0 + \lambda_1 + \cdots + \lambda_j$.

Secondly, reversing the projection we see that $1+2+7$ could lift to $1+2+7+15$ but could also lift to a Bachet partition $1+2+7+16$ of 26. But what we can say is the following: if $m' = \lambda_0 + \lambda_1 + \cdots + \lambda_{n-1}$ is a Bachet partition of m' (implicit in this statement is that $n-1 = \lfloor \log_3(2m') \rfloor$) then we can extend it to a Bachet partition $\lambda_0 + \lambda_1 + \cdots + \lambda_{n-1} + (m - m')$ of a fixed m if and only if

$$\text{(i) } \lambda_{n-1} \le m - m', \text{ (ii) } m \text{ has the property that } n = \lfloor \log_3(2m) \rfloor$$
$$\text{and (iii) } m - m' \le 1 + 2m' \text{ or } \lceil \tfrac{m-1}{3} \rceil \le m'.$$

In the case of the Bachet partitions for $m = 25$ below, the boldfaced largest terms are peeled off to leave precisely the Bachet partitions for $m' = 8, 9, 10, 11, 12$ and 13 and, by the projections of the hereditary property, no other Bachet partitions can be built upon to provide Bachet partitions for $m = 25$.

$$\begin{aligned} 25 &= 1+3+9+\mathbf{12} &&= 1+3+8+\mathbf{13} &&= 1+3+7+\mathbf{14} \\ &= 1+3+6+\mathbf{15} &&= 1+3+5+\mathbf{16} &&= 1+3+4+\mathbf{17} \\ &= 1+2+7+\mathbf{15} &&= 1+2+6+\mathbf{16} &&= 1+2+5+\mathbf{17} \end{aligned}$$

In other words, letting Bachet(m) denote the set of Bachet partitions of m,

$$|\text{Bachet}(25)| \;=\; \sum_{m'=8}^{13} |\text{Bachet}(m')| = 9 = f(40-25) = f(15)$$

We claim that $|\text{Bachet}(m)| = f(\frac{1}{2}(3^{n+1}-1)-m)$ holds whenever m is sandwiched by

$$\frac{1}{2}(3^n-1)+3^{n-1} \leq m \leq \frac{1}{2}(3^{n+1}-1).$$

Let's refer to such m's as simply being *sandwiched*. For sandwiched m's, condition (ii) above is immediately taken care of. We already have $m' \leq \frac{1}{2}(3^n-1)$ and $\lambda_{n-1} \leq 3^{n-1}$ which implies that $m - m' \geq (\frac{1}{2}(3^n-1)+3^{n-1}) - \frac{1}{2}(3^n-1) = 3^{n-1} \geq \lambda_{n-1}$ as needed for condition (i). And in order that condition (iii) is met we need simply to insist that those Bachet(m')'s that extend to Bachet partitions of m are exactly those in the range $\lceil \frac{m-1}{3} \rceil \leq m' \leq \frac{1}{2}(3^n-1)$. Hence

$$\text{Bachet}(m) \;=\; \bigcup_{m'=\lceil \frac{m-1}{3} \rceil}^{\frac{1}{2}(3^n-1)} \text{Bachet}(m').$$

By the projecting and lifting of the hereditary property, each Bachet partition of m' is extended to a *unique* Bachet partition of m. Hence, the number of elements in the above union equals the sum of the number of elements in each Bachet(m') of that union. So whenever m is sandwiched we have

$$|\text{Bachet}(m)| \;=\; |\bigcup_{m'=\lceil \frac{m-1}{3} \rceil}^{\frac{1}{2}(3^n-1)} \text{Bachet}(m')| \;=\; \sum_{m'=\lceil \frac{m-1}{3} \rceil}^{\frac{1}{2}(3^n-1)} |\text{Bachet}(m')|.$$

We are no ready to tie together Bachet partitions and ternary partitions. We claim that $f(\frac{1}{2}(3^{n+1}-1)-m) = |\text{Bachet}(m)|$ for all sandwiched m's. We do so once again by induction on $n = \lfloor \log_3(2m) \rfloor$. The claim holds for $n=1$ since $2 = 1+1$; $3 = 1+2$; $4 = 1+3$ and also for all sandwiched m's for $n=2$, as can be seen here:

$$7 = 1+3+3 = 1+2+4 = 1+1+5;\; 8 = 1+3+4 = 1+2+5;\; 9 = 1+3+5 = 1+2+6;$$
$$10 = 1+3+6 = 1+2+7;\; 11 = 1+3+7;\; 12 = 1+3+8;\; 13 = 1+3+9.$$

So assume that m is sandwiched with $n = \lfloor \log_3(2m) \rfloor$. We already know that $|\text{Bachet}(m)| = \sum_{m'=\lceil \frac{m-1}{3} \rceil}^{\frac{1}{2}(3^n-1)} |\text{Bachet}(m')|$ and we remark that every such m' in this summation is also sandwiched, but with $n-1 = \lfloor \log_3(2m') \rfloor$. Hence, by our inductive hypothesis, $f(\frac{1}{2}(3^n-1)-m') = |\text{Bachet}(m')|$ and

$$|\text{Bachet}(m)| \;=\; \sum_{m'=\lceil \frac{m-1}{3} \rceil}^{\frac{1}{2}(3^n-1)} f(\frac{1}{2}(3^n-1)-m') \;=\; f(0)+f(1)+f(2)+\cdots f(\frac{1}{2}(3^n-1)-\lceil \frac{m-1}{3} \rceil).$$

But the input of the last term $\frac{1}{2}(3^n-1) - \lceil \frac{m-1}{3} \rceil$ simplifies to $\lfloor \frac{\frac{1}{2}(3^{n+1}-1)-m}{3} \rfloor$ and so we have

$$|\text{Bachet}(m)| \;=\; \sum_{i=0}^{\lfloor \frac{\frac{1}{2}(3^{n+1}-1)-m}{3} \rfloor} f(i)$$

This is exactly the recurrence relation, with the initial conditions still intact, that we had hoped to obtain. Hence when m is sandwiched, the generating function for the Bachet partitions is exactly $F(x)$, the generating function for the ternary partitions.

We close this section by briefly remarking on those m's that are not sandwiched. Using the generating function $F(x)$ we can define another

$$G(x) := \sum_{k=0}^{\infty} g(k)x^k \; = \sum_{j=0}^{\infty} \frac{x^{3^j}-1}{1-x^{2(3^j)}} F(x^{5(3^j)}) \prod_{i=0}^{j} \frac{1}{1-x^{3^i}}.$$

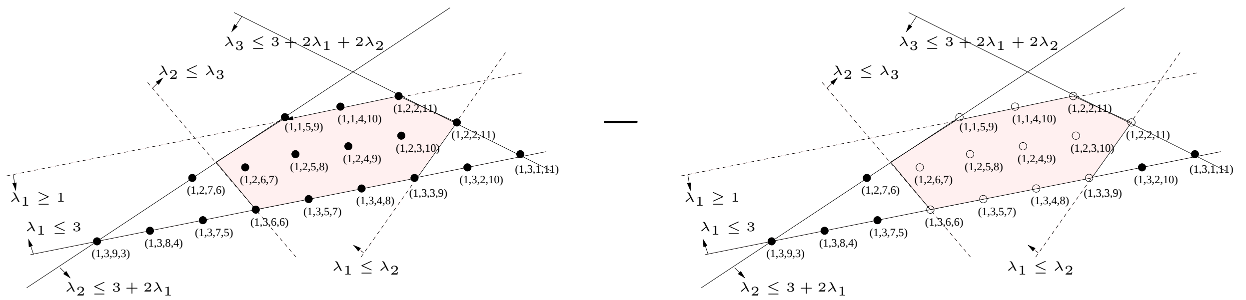
Then, adapting the convention that $g(k) = 0$ if k is a negative integer, Rødseth's formula claims that the number of Bachet partitions of m equals

$$f(\frac{1}{2}(3^{n+1}-1)-m) - g((\frac{1}{2}(3^n-1)+3^{n-1}-1)-m)$$

Note that m is sandwiched precisely when the input for $g(\cdot)$ is negative. An example of a non-sandwiched m is 16 and for this we have $|\mathrm{Bachet}(16)| = f(24) - g(5) = 18 - 6$ (we only have to work out the first two parts $j = 0, 1$ of the infinite sum for $g(5) = 0$). We can list the Bachet partitions for 16, using Theorem 5:

$$\begin{aligned} 16 &= 1+3+3+9 &&= 1+3+4+8 &&= 1+3+5+7 &&= 1+3+6+6 \\ &= 1+2+6+7 &&= 1+2+5+8 &&= 1+1+5+9 &&= 1+2+4+9 \\ &= 1+1+4+10 &&= 1+2+3+10 &&= 1+2+2+11 &&= 1+1+3+11 \end{aligned}$$

Sticking to our promise not to prove Rødseth's formula we will instead present a polyhedral picture as to why the generating function $G(x)$ is needed when m is not sandwiched. In the previous section we saw a figure showing the Bachet partitions for $m = 25$ as lattice points in a triangle. A key observation here, and something that holds for all sandwiched m's, is that the ordering inequalities $\lambda_0 \le \lambda_1 \le \cdots \le \lambda_n$ were subsumed by the $\lambda_i \le 1 + 2(\lambda_0 + \cdots + \lambda_{i-1})$. This is a significant part of what made counting Bachet(m) for sandwiched m's relatively straightforward: we get the ordering of the parts for free from the other inequalities! Not so when m is not sandwiched and we can see this in the following figure for $m = 16$:



The 18 points on the left indicate the lattice points in the polyhedron defined by $\lambda_1 \le 3$, $\lambda_2 \le 3+2\lambda_1$ & $\lambda_3 \le 3+2\lambda_1+2\lambda_2$ and living in the plane defined by $\lambda_0 = 1$ and $\lambda_0+\lambda_1+\lambda_2+\lambda_3 = 16$. This polyhedron pays no regard to the ordering of the parts in the Bachet partitions. This is precisely what $f(40-24) = 18$ is counting. On the right we see the effect of including the inequalities that define the ordering on the parts $\lambda_0 \le \lambda_1 \le \lambda_2 \le \lambda_3$: we need to subtract exactly $g(5) = 6$ lattice points from those on the left to get the count of $|\mathrm{Bachet}(16)| = 12$ just right!

We motivated why Rødseth's general formula works by explaining it in terms of counting lattice points in polyhedra. This would not be the usual approach in integer partitions owing to the fact that partitions can rarely be described in terms of lattice points in a polyhedron. Many of the wonderful results in integer partitions depend largely on the ability to manipulate generating functions in much the same way that we attained the functional equation $(1-x)F(x) = F(x^3)$. However, it is no fluke that the generating functions above counted the lattice points and this method of counting lattice points in a polyhedron (by generating functions) is one of the most beautiful and effective methods for solving the general problem of counting lattice points in polyhedra. Readers who are interested in finding out more about this remarkable area of active study can see [6, Ch.9, *Brion's formula*]. The work of Barvinok & Woods [5] shows how generating function methods are not only beautiful but surprisingly computationally efficient.

THE ONE-SCALE BACHET'S PROBLEM

A natural variant of Bachet's problem is the one-scale problem: *what is the least number of pound weights that can be used on a scale pan to weigh any integral number of pounds from 1 to m inclusive, if the weights can be placed in only one of the scale pans ?* This problem is completely understood (and proven) in much the same manner as the two-scale problem where powers of 3 are replaced with powers of 2. For completeness and posterity's sake, let's briefly document the results here.

Call a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ a *one-scale Bachet partition* if every integer $0 \leq l \leq m$ can be written as $l = \sum_{i=0}^{n} \beta_i \lambda_i$ where each $\beta_i \in \{0, 1\}$ and furthermore there does not exist another partition of m satisfying this condition with fewer parts than $n+1$. In much the same spirit as Proposition 1 for every positive integer m there is a unique n such that $2^n \leq m \leq 2^{n+1} - 1$. With this corresponding n we can form the partition $m = 1+2+4+\cdots+2^{n-1}+(m-2^n+1)$ with a possible reordering of the parts if needed. Lemma 2 can be tweaked to show that every one-scale Bachet partition satisfies $\lambda_0 = 1$ and for every $i = 1, 2, \ldots, n$ we have $\lambda_i \leq 1 + 1(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$. This implies in turn that $\lambda_i \leq 2^i$ and that the minimality of parts condition for the one-scale problem reduces to $n := \lfloor \log_2(m) \rfloor$. As in the two-scale problem we have the following result, whose proof is very similar to that of Theorem 5:

Theorem 6. *The partition $m = \lambda_0 + \lambda_1 + \lambda_2 + \cdots + \lambda_n$ is a one-scale Bachet partition if and only if $n = \lfloor \log_2(m) \rfloor$, $\lambda_0 = 1$ and $\lambda_i \leq 1 + (\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for every $i = 1, 2, \ldots, n$.*

The above theorem, without the minimality of parts condition was first shown by Brown [8] who called them *complete partitions* and the minimality condition was noted by Park [16]. We called the one-scale Bachet partitions *M-partitions* [15], being unaware of Brown's and Park's contributions at the time of writing [15]. We hope that their work's inclusion here provides them with their delinquent due. There are six one-scale partitions for 25:

$$\begin{aligned} 25 &= 1+2+4+8+10 &&= 1+2+4+7+11 &&= 1+2+4+6+12 \\ &= 1+2+3+7+12 &&= 1+2+4+5+13 &&= 1+2+3+6+13 \end{aligned}$$

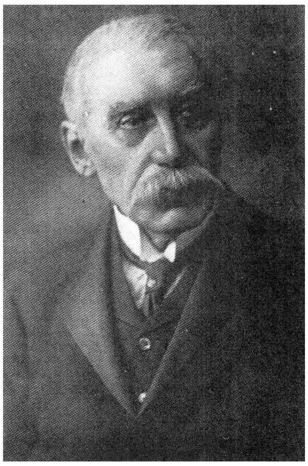
In [15] a recurrence relation was derived for the one-scale Bachet partitions. In this context, m is *sandwiched* if it satisfies $2^n - 1 + 2^{n-1} \leq m \leq 2^{n+1} - 1$. The number of one-scale Bachet partitions for a sandwiched m equals $f_1(2^{n+1} - 1 - m)$ where

$$f_1(k) = \sum_{i=0}^{2^n - 1 - \lceil \frac{m-1}{2} \rceil} f_1(i)$$

with the initial condition of $f_1(0) = 1$. From this recurrence we get $f_1(31-25) = f_1(6) = 1+1+2+2$ as expected from the six partitions of 25 listed above. The corresponding generating function is in turn that of the *binary partitions*. The general formula is once again due to Rødseth [18] and is a generating function formula analogous to the case in the previous section.

MacMahon's perfect partitions

In 1886 Major Percy A. MacMahon [12] [14, pp. 217–223] proposed and solved an alternative generalization to Bachet's problem, which differs significantly from the generalization that we have investigated until now. It's appropriate too that we should include MacMahon's contribution to Bachet's problem since, as Gian-Carlo Rota persuasively argues in his introduction to MacMahon's collected papers [13], MacMahon's substantial contributions to the foundation of modern combinatorics have not always been given their proper due.



MacMahon noted that the example of $40 = 1 + 3 + 9 + 27$ had the property that every integer weight l between 1 and 40 can be weighed in a *unique* manner using the weights $1, 3, 9$ and 27 on a two-scale pan. In other words, MacMahon discarded the *minimality of parts* condition that we focused on here and instead added the *uniqueness* condition. The set of all partitions for MacMahon's generalization of Bachet's problem for 40 is thus: $\underbrace{1+1+\cdots+1+1}_{40\text{ times}}$, $\underbrace{1+1+\cdots+1+1}_{13\text{ times}}+27$, $1+\underbrace{3+3+\cdots+3+3}_{13\text{ times}}$, $1+1+1+1+9+9+9+9$, $1+1+1+1+9+27$, $1+3+3+3+3+27$, $1+3+9+9+9+9$, $1+3+9+27$. For shorthand, we write these partitions respectively as

$$(1)^{40},\ (1)^{13}+27,\ 1+(13)^3,\ (1)^4+(9)^4,\ (1)^4+9+27,\ 1+(3)^4+27,\ 1+3+(9)^4,\ 1+3+9+27.$$

Note that we view the repeated parts of such partitions as indistinguishable as in the case of $(1)^{13} + 27$ we regard there being a unique expression for 4 as $1 + 1 + 1 + 1$; not $\binom{13}{4}$ distinct expressions! Also note that it includes our unique Bachet partition for $m = 40$. To describe all such partitions it will be easier to begin by analyzing the one-scale analogue of MacMahon's generalization, from which the two-scale problem will follow immediately.

Starting with the one-scale problem, MacMahon called a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_s$ *perfect* if every l between 0 and m can be written uniquely as $l = \sum_{i=0}^{s} \alpha_i \lambda_i$ for $\alpha_i \in \{0, 1\}$ (and with repeated parts regarded as indistinguishable as described above). For example, we have 8 perfect partitions for $m = 11$:

$$(1)^{11},\ (1)^5+(6)^1,\ (1)^1+(2)^5,\ (1)^3+(4)^2,\ (1)^2+(3)^2,\ (1)^2+(3)^1+(6)^1,\ (1)^1+(2)^2+(6)^1,\ (1)^1+(2)^1+(4)^2$$

MacMahon's insight was that the perfect partitions of m are in bijection with the *ordered factorizations* of the integer $m + 1$ – the set of all possible factorizations (not including multiples of 1) of $m + 1$ but where we account for order too. For example, the set of ordered factorizations for 12 equals

$$12,\ 6\times 2,\ 2\times 6,\ 4\times 3,\ 3\times 4,\ 3\times 2\times 2,\ 2\times 3\times 2,\ 2\times 2\times 3.$$

Theorem 7. (MacMahon [12]) *The perfect partitions of m are in bijection with the ordered factorizations of $m + 1$.*

Proof. Consider the ordered factorization $f_1 \times f_2 \times f_3 \times \cdots \times f_r$ of $m+1$. From this factorization consider the recursively defined partition of m:

$$(1)^{f_1-1}+(f_1)^{f_2-1}+(f_1\cdot f_2)^{f_3-1}+(f_1\cdot f_2\cdot f_3)^{f_4-1}+\cdots+(f_1\cdot f_2\cdots f_{i-1})^{f_i-1}+\cdots+(f_1\cdot f_2\cdots f_{r-1})^{f_r-1}$$

This partition sums to m since the parts form a telescopic sum which collapses to $f_1\cdot f_2\cdot f_3\cdots f_r-1$.

To see that the partition is perfect note that the parts $(1)^{f_1-1}$ suffice to expresses every $1 \le l_1 \le f_1 - 1$. Since all other parts of the partition are larger than f_1 then the sum of l_1 1's is the unique way to express l_1 using parts of the above partition. Next, each $l_2 = f_1, 2f_1, \ldots, (f_1-1)(f_2)$ can be expressed by the parts $(f_1)^{f_2-1}$. And, as argued above, using the parts $(1)^{f_1-1}$ we can express every integer less than or equal to $(f_1 \cdot f_2 - 1)$ uniquely as $l_1 + l_2$ where l_1 and l_2 are one of those above. Since the other parts of the partition are larger than $f_1 \cdot f_2 - 1$ then these are the unique expressions for every integer less than or equal to $f_1 \cdot f_2 - 1$. Now repeat the argument; rigorously one needs to complete the argument by induction on the number of terms in the ordered factorization.

With much the same argument in mind, it's not too difficult to see that any perfect partition $m = (\lambda_0)^{g_1-1} + (\lambda_1)^{g_2-1} + (\lambda_2)^{g_3-1} + \cdots + (\lambda_{s-1})^{g_s-1}$ must provide an ordered factorization of $m+1$: there must be at least one part of the partition equal to 1 since we must be able to express 1 as a subsum of the parts of the partition. Hence $\lambda_0 = 1$. Next $g_1 = \lambda_1$: if $\lambda_1 < g_1$ then the integer $\lambda_1 = g_1 + (1)^{\lambda_1 - g_1} = (1)^{\lambda}_1$ which would upset the uniqueness property. And if $\lambda_1 > g_1$ then we would have no way of expressing g_1 as a subsum of the parts of the partition. Hence, $\lambda_1 = g_1$. Assuming then that λ_1 is repeated $g_2 - 1 \ge 1$ times we are then, by a similar argument, forced to have $\lambda_2 = g_1 \cdot g_2$ and we can repeat this argument to yield $\lambda_i = g_1 \cdot g_2 \cdots g_i$ and this perfect partition yields an ordered factorization of $m+1$ vis-a-vis $g_1 \times g_2 \times \cdots \times g_s$. □

The perfect partitions for 11 and the unique factorizations of 12 listed above are done so in the order of the bijection between the sets. For example $12 \leftrightarrow (1)^{12-1} = 1 + 1 + \cdots + 1$ and $2 \times 3 \times 2 \leftrightarrow (1)^{2-1} + (2)^{3-1} + (2 \cdot 3)^{2-1} = 1 + 2 + 2 + 3 + 3$. We can't say for certain but would be willing to wager that MacMahon's motivation for calling these partitions "perfect" would be that the confluence between factorizations and sums reminded him of a similar confluence seen in perfect numbers.

With this characterization of perfect partitions we can in turn solve the *two-scale* problem – these are what MacMahon called *subperfect partitions*. MacMahon called a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_s$ *subperfect* if every l between 0 and m can be written uniquely as $l = \sum_{i=0}^{s} \alpha_i \lambda_i$ for $\alpha_i \in \{-1, 0, 1\}$ (and with repeated parts regarded as indistinguishable as in the case of perfect partitions).

Theorem 8. [12, §3] *The ordered factorizations of* $2m+1$ *are in bijection with the subperfect partitions of* m.

Proof. Consider any ordered factorization of $2m + 1 = f_1 \times f_2 \times f_3 \times \cdots \times f_r$. Since $2m+1$ is odd then each $f_i \ge 3$ and must also be odd. In turn, each $f_i - 1 \ge 2$ and is even. From Theorem 7 we have a perfect partition of $2m$ given by

$$(1)^{f_1-1}+(f_1)^{f_2-1}+(f_1\cdot f_2)^{f_3-1}+(f_1\cdot f_2\cdot f_3)^{f_4-1}+\cdots+(f_1\cdot f_2\cdots f_{i-1})^{f_i-1}+\cdots+(f_1\cdot f_2\cdots f_{r-1})^{f_r-1}$$

By definition, every l between 0 and $2m$ can be expressed in a unique way as a subsum of these parts. In other words, as a $\{0,1\}$-combination of the parts of this perfect partition. However, each one of the $f_i - 1$'s are even and so every indistinguishable part appears an even number of times in the partition and so we can rephrase the above partition being a perfect partition for $2m$ as

$$(1)^{\frac{f_1-1}{2}}+(f_1)^{\frac{f_2-1}{2}}+(f_1\cdot f_2)^{\frac{f_3-1}{2}}+(f_1\cdot f_2\cdot f_3)^{\frac{f_4-1}{2}}+\cdots+(f_1\cdot f_2\cdots f_{i-1})^{\frac{f_i-1}{2}}+\cdots+(f_1\cdot f_2\cdots f_{r-1})^{\frac{f_r-1}{2}}$$

is a 2-complete partition of m with MacMahon's uniqueness property preserved. But as we have noted in earlier sections, 2-complete partitions are exactly the Bachet partitions without the minimality of parts constraint. Since the uniqueness property is also preserved then we have shown that the subperfect partitions of m are given precisely by the ordered factorizations of $2m+1$. □

The eight subperfect partitions of 40 that we opened this section with are attained respectively from the ordered factorizations of 81:

$$81,\ 27\times 3,\ 3\times 27,\ 9\times 9,\ 9\times 3\times 3,\ 3\times 9\times 3,\ 3\times 3\times 9,\ 3\times 3\times 3\times 3.$$

All in all, we get the MacMahon's two-scale problem for (almost) free from the one-scale problem and the connection to between (additive) partitions and (multiplicative) factorizations is surprising and satisfying. As claimed in the introduction, the original Bachet partition of $40 = 1+3+9+27$ comes from the factorization $3\times 3\times 3\times 3$ of 81.

Closing Remarks

The r-complete partitions The 1-complete and 2-complete partitions discussed here fit a general definition of a partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ being *r-complete* if every integer $0 \le l \le rm$ can be written as $l = \sum_{i=0}^{n} \alpha_i \lambda_i$ where each $\alpha_i \in \{0, 1, 2, \ldots, r\}$. Park [16] showed that a partition is r-complete if and only if $\lambda_i \le 1 + r(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for each $i = 1, 2, \ldots, n$. As we have discussed here, the r-complete partitions have a natural interpretation as a physical process in terms of scale pans when $r = 1$ and $r = 2$. We would like to know if there is any satisfactory physical interpretation for r-complete partitions when $r \ge 3$.

The relaxed Bachet's problem Like the original Bachet's problem we start with a puzzle. *We are given a two-scale balance and a bag of flour that has an integer valued weight l between one and eight ounces. Given that the weights can be placed in either of the pans, what is the least number of ounce weights that can be used to discern the integer weight l of our bag of flour ?*

The partition $8 = 2 + 6$ will suffice since if the weight l placed on the right scale were equal to 2, 4, 6 or 8 then this would result in a balanced scale when 2, $6-2$, 6 or $2+6$ respectively are placed on the two-scale balance (as before, positive coefficients are assigned to weights placed on the left scale, negative to those on the right). As for the remaining possible l's 1 is lighter than 2 but heavier than nothing; 3 is lighter than 4 but heavier than 2 and so 3 can be discerned using the parts of the partition $8 = 2 + 6$. Similarly for $6 - 2 < 5 < 6$ and $6 < 7 < 8$.

The above discussion can be formally described as follows: find a partition of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ such that n is as small as possible with the property that no two consecutive integers between 1 and m are absent from the set $\{\sum_{i=0}^{n} \beta_i \lambda_i : \beta_i \in \{-1, 0, 1\}\}$. This is the right definition to mirror the problem above since if two consecutive integers are absent from the $\{-1, 0, 1\}$-combinations of $\lambda_0, \lambda_1, \ldots, \lambda_n$ then we have no way of distinguishing between these two integers. Call such a partition a *relaxed Bachet partition of m*. Note that $8 = 2 + 6$ is a relaxed Bachet partition for 8 since the only partition of 8 with one part is 8 itself and we need more than this to distinguish all integers less than 8.

We can understand relaxed Bachet partitions with much the same reasoning that we used to understand Bachet partitions. One immediate candidate for a relaxed Bachet partition comes from doubling all parts in a Bachet partition. For example, $8 = 2 + 6$ was attained from doubling the parts of $4 = 1 + 3$. Since $1 + 3$ can weigh every integer between 1 and 4 then doubling the terms to $2 \cdot 1$ and $2 \cdot 3$ ensures that every even integer between 1 and 8 can be discerned from 2 and 6, as argued at the beginning of this section. In general, we have $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a relaxed Bachet partition if and only if $n = \lfloor \log_3(m) \rfloor$, $\lambda_0 \leq 2$ and $\lambda_i \leq 2 + 2(\lambda_0 + \cdots + \lambda_{i-1})$ for each $i = 1, 2, \ldots, n$. This will be expounded upon further in [9].

Connections to other areas Permit us to finish this article not so much with a criticism of Bachet partitions but by expressing a yearning. While the Bachet partitions span from a highly accessible problem with their general solutions both elegant and succinct, we know of no other connection to other seemingly unrelated areas of mathematics, even on the basic level like that which takes place between perfect partitions and factorizations.

To expound on this further, a distinguishing feature of Bachet partitions is that they can be described in terms of inequalities on the parts, a feature shared by the *lecture hall partitions* of Bousquet-Mélou & Eriksson [7]: given N rows in a lecture hall we wish for the height of each row (the λ_i's) to be in such a way that from every seat one has a clear view of the speaker. Formally a lecture hall partition of length N is a partition with N parts $\lambda_1 + \lambda_2 + \cdots \lambda_N$ such that $0 \leq \frac{\lambda_1}{1} \leq \frac{\lambda_2}{2} \leq \cdots \leq \frac{\lambda_N}{N}$. The context for these partitions first arose in the algebraic setting of *Coxeter groups* and from this setting one gets rich results like [3, §9.3]: *the number of lecture hall partitions of length N equals the number of partitions in which all parts are odd and less than* $2N$. Moreover, no direct proof of this lecture hall theorem is known and was only made possible by delving into the parallel world of Coxeter groups. Without such connections between Bachet partitions and other areas, there may be beautiful structural results for Bachet partitions that are easily transparent when seen in other contexts, and conversely, the results we know on Bachet partitions may shed light on these other areas. Finding such a connection, if there is one, is highly desirable.

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