Integration by parts

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MA100

Integration by parts (indefinite version)

Theorem

Suppose f is a differentiable function and g is a continuous functions, and that $G(x)$ is an antiderivative of g, then

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

Motivation

$$\frac{d}{dx}\left(f(x)\,G(x)\right) = f'(x)G(x) + f(x)g(x)$$

so

$$\int f(x)g(x) + f'(x)G(x)\,dx = f(x)\,G(x)$$

$$\int f(x)g(x)\,dx + \int f'(x)G(x)\,dx = f(x)\,G(x)$$

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

Example

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$$

Find

$$\int \underset{\downarrow}{x} \overset{\uparrow}{\sin x}\, dx$$

$\downarrow$: differentiate

$\uparrow$: integrate

$$\int \underset{\downarrow}{x} \cdot \overset{\uparrow}{\sin x}\, dx = \begin{bmatrix} f(x) = x \\ g(x) = \sin x \end{bmatrix} - x \cdot \cos x - \int 1 \cdot (-\cos x)\,dx$$

$f(x) = x$

$G(x) = -\cos x$

$f'(x) = 1$

$$= -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C$$

We can check that it is correct:

$$\frac{d}{dx}\left(-x\cos x + \sin x + C\right) = -\cos x + x\sin x + \cos x$$

$$= x \sin x.$$

Example

Still $\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx$

Find
$$\int \ln x \, dx$$

$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx$$

so $f(x) = \ln x$, $g(x) = 1$
$f'(x) = \frac{1}{x}$, $G(x) = x$

$$= \underbrace{x}_{G(x)}\,\underbrace{\ln x}_{f(x)} - \int \underbrace{\frac{1}{x}}_{f'(x)} \cdot \underbrace{x}_{G(x)}\, dx = x \ln x - \int 1 \, dx =$$

$$= x \ln x - x + C$$

Example

$\int f(x)g(x) = f(x)\,G(x) - \int f'(x)G(x)\,dx$

Find

$$\int t^2 e^t \, dt$$

$\int t^2 e^t \, dt = \quad f(t) = t^2 \quad g(t) = e^t \quad t^2 e^t - \int 2t e^t \, dt$

$f'(t) = 2t \quad G(t) = e^t$

We now use integration by parts a second time,

with $f(t) = 2t \quad g(t) = e^t \quad = t^2 e^t - \left(2t e^t - \int 2 e^t \, dt \right)$

$f'(t) = 2 \quad G(t) = e^t$

$= t^2 e^t - 2t e^t + \int 2 e^t \, dt = t^2 e^t - 2t e^t + 2e^t + C$

Check: $\frac{d}{dt}\left(t^2 e^t - 2t e^t + 2e^t + C \right) = t^2 e^t + 2t e^t - 2t e^t - 2e^t + 2e^t = t^2 e^t$

Example

$\int f(x)\, g(x)\, dx = f(x)\, G(x) - \int f'(x)\, G(x)\, dx$

Find

$$\int e^x \sin x \, dx$$

$\int e^x \sin x \, dx =$

$f(x) = e^x \quad g(x) = \sin x$

$f'(x) = e^x \quad G(x) = -\cos x$

$-e^x \cos x + \int e^x \cos x \, dx$

Use integration by parts again with $f(x) = e^x$, $f'(x) = e^x$, $g(x) = \cos x$

$G(x) = \sin x$

$-e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx \right) =$

$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$

This gives: $2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x \right) + C,$$

Integration by parts (definite version)

Theorem

Suppose f is a differentiable function and g is a continuous functions, and that $G(x)$ is an antiderivative of g, then

$$\int_a^b f(x)g(x)\,dx = [f(x)G(x)]_a^b - \int_a^b f'(x)G(x)\,dx$$

This is a direct consequence of the indefinite version of theorem + F.T.C (Fundamental Theorem of Calculus)

Example

$$\int_a^b f(x)g(x)\,dx = \Big[f(x)G(x)\Big]_a^b - \int_a^b f'(x)G(x)\,dx$$

Calculate

$$\int_0^1 \tan^{-1} x\, dx$$

We view the integrand $\tan^{-1} x$ as a product:

$$\tan^{-1} x = 1 \cdot \tan^{-1} x$$

and use integration by parts:

$$\int_0^1 \tan^{-1} x\, dx = \int_0^1 1 \cdot \tan^{-1} x\, dx =$$

$$= \Big[x \tan^{-1} x\Big]_0^1 - \int_0^1 \frac{x}{1+x^2}\, dx$$

$f(x) = \tan^{-1} x$

$f'(x) = \dfrac{1}{1+x^2}$

$g(x) = 1$

$G(x) = x$

Example (cont.)

$$\left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \underbrace{1\cdot\tan^{-1}1}_{\pi/4} - \underbrace{0\cdot\tan^{-1}0}_{0} - \underbrace{\int_0^1 \frac{x}{1+x^2}\,dx}_{\frac{1}{2}\ln 2}$$

Here: $1\cdot\tan^{-1}1 = \frac{\pi}{4}$

$0\cdot\tan^{-1}0 = 0$

$$= \frac{\pi}{4} - \frac{1}{2}\ln 2$$

$$\int_0^1 \frac{x}{1+x^2}\,dx =$$

Change variables:

$u = 1+x^2$

$\frac{du}{dx} = 2x \quad du = 2x\,dx$

$x=0 \Rightarrow u=1$

$x=1 \Rightarrow u=2$

$$= \frac{1}{2}\int_0^1 \frac{1}{1+x^2}\,2x\,dx = \frac{1}{2}\int_1^2 \frac{1}{u}\,du = \frac{1}{2}\left[\ln u\right]_1^2 = \frac{1}{2}\ln 2 - \frac{1}{2}\ln 1$$

$$= \frac{1}{2}\ln 2$$