Trigonometric integrals and substitutions

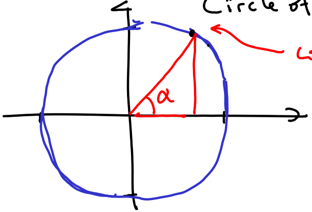
E. Sköldberg
emil.skoldberg@nuigalway.ie
http://www.maths.nuigalway.ie/~emil/

School of Mathematics etc.
National University of Ireland, Galway

MA100

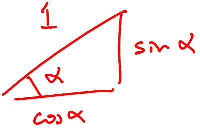
Trigonometry

Circle of radius 1

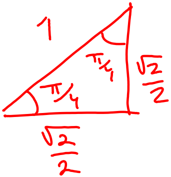


coordinates $(\cos\alpha, \sin\alpha)$

so $\boxed{\sin^2\alpha + \cos^2\alpha = 1}$



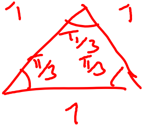
Important angles: $30^\circ = \frac{\pi}{6}$, $45^\circ = \frac{\pi}{4}$, $60^\circ = \frac{\pi}{3}$, $90^\circ = \frac{\pi}{2}$.



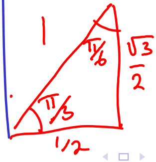
so

$\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$

$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$



cut in half



so

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$\cos\frac{\pi}{3} = \frac{1}{2}$

$\sin\pi/6 = 1/2$

$\cos\pi/6 = \frac{\sqrt{3}}{2}$

Trigonometric Identities

Addition:

$$\sin(x+y) = \sin x \cos y + \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

so

$$\sin 2x = \sin(x+x) = 2 \sin x \cos x$$

$$\cos 2x = \cos(x+x) = \cos^2 x - \sin^2 x$$

$$\left(= 2\cos^2 x - 1\right) \quad \left(= 1 - 2\sin^2 x\right)$$

Odd power of cos

Evaluate

$$\int \cos^3 x \, dx$$

Save one power of cos, convert the rest to sin:

$$\int \cos^3 x \, dx = \int (1-\sin^2 x)\cos x \, dx = \left[\begin{array}{l} u = \sin x \\ \frac{du}{dx} = \cos x \\ du = \cos x \, dx \end{array} \right]$$

$$= \int (1-u^2)\, du = u - \frac{u^3}{3} + C$$

$$= \sin x - \frac{\sin^3 x}{3} + C$$

Odd power of sin

Find
$$\int \sin^3 x \cos^4 x \, dx$$

Save one power of sin, convert the rest to cos:

$$\int \sin^3 x \cos^4 x \, dx = \int (1-\cos^2 x)\sin x \cdot \cos^4 x \, dx =$$

$$= \int (1-\cos^2 x)\cos^4 x \sin x \, dx = \left[\begin{array}{l} u = \cos x \\ \frac{du}{dx} = -\sin x \quad du = -\sin x \, dx \end{array}\right]$$

$$= -\int (1-u^2)u^4 \, du = \int u^6 - u^4 \, du = \frac{u^7}{7} - \frac{u^5}{5} + C$$

$$= \frac{\cos^7 x}{7} - \frac{\cos^5 x}{5} + C$$

Even power of sin and cos

Find

$$\int \sin^2 x \, dx$$

We saw earlier that $\cos 2x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$

So $\cos^2 x = \frac{1+\cos 2x}{2}$ and $\sin^2 x = \frac{1-\cos 2x}{2}$

Using this, we get:

$$\int \sin^2 x \, dx = \int \frac{1-\cos 2x}{2} dx = \frac{1}{2}\int 1 - \cos 2x \, dx =$$

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C = \frac{x}{2} - \frac{\sin 2x}{4} + C$$

Even power of sin and cos

Find

$$\int \sin^2 x \cos^2 x \, dx$$

We will use the identities $\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

$$\int \sin^2 x \cos^2 x \, dx =$$

$$\int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} dx = \frac{1}{4} \int 1 - \cos^2 2x \, dx =$$

$$= \frac{1}{4} \int 1 - \frac{1+\cos 4x}{2} dx = \frac{1}{8} \int 1 - \cos 4x \, dx = \frac{x}{8} - \frac{\sin 4x}{32} + C$$

Strategy

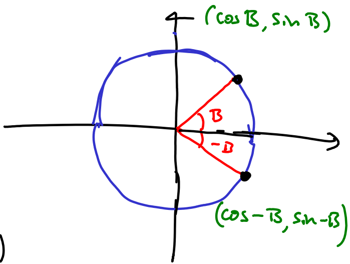
To evaluate

$$\int \sin^m x \cos^n x \, dx$$

- If m is odd, save one power of sin and convert the rest to cos using $\sin^2 x = 1 - \cos^2 x$
- If n is odd, save one power of cos and convert the rest to sin using $\cos^2 x = 1 - \sin^2 x$
- If both n and m are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

More useful identities



We have for all angles A and B:

- $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$
- $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$
- $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$

$\Rightarrow \cos(-B) = \cos B$

$\sin(-B) = -\sin B$

We have

$\underline{\sin(A+B)} = \underline{\sin A \cos B + \cos A \sin B}$

(if $B = -B$)

$\underline{\sin(A-B)} = \sin A \cos(-B) + \cos A \sin(-B) = \underline{\sin A \cos B - \cos A \sin B}$

&

$\underline{\cos(A+B)} = \underline{\cos A \cos B - \sin A \sin B}$

so $\underline{\cos(A-B)} = \cos A \cos(-B) - \sin A \sin(-B) = \underline{\cos A \cos B + \sin A \sin B}$

From these we can deduce the formulas

To evaluate integrals of the form

$$\int \sin mx \cos nx \, dx$$

we can use the identities from the previous slide.

Example

Evaluate

$$\int \sin 4x \cos 5x \, dx$$

To find the antiderivative, we use the formula

$$\sin A \cos B = \frac{1}{2}\left(\sin(A-B) + \sin(A+B)\right)$$

with $A = 4x$ $B = 5x$, so

$$\sin 4x \cos 5x = \frac{1}{2}\left(\sin(4x-5x) + \sin(4x+5x)\right)$$

$$= \frac{1}{2}\left(\sin(-x) + \sin 9x\right) = \frac{1}{2}\left(\sin 9x - \sin x\right)$$

Using this identity, we get:

$$\int \sin 4x \cos 5x \, dx = \frac{1}{2}\int \sin 9x - \sin x \, dx = -\frac{\cos 9x}{18} + \frac{\cos x}{2} + C$$

Example

Evaluate

$$I := \int \sin 4x \cos 5x \, dx$$

We could also use integration by parts:

$$\int \sin 4x \cdot \cos 5x \, dx = \sin 4x \cdot \frac{\sin 5x}{5} - \int 4 \cos 4x \cdot \frac{\sin 5x}{5} \, dx$$

$$= \frac{1}{5} \sin 4x \sin 5x - \frac{4}{5} \int \cos 4x \cdot \sin 5x \, dx$$

$$= \frac{1}{5} \sin 4x \sin 5x - \frac{4}{5} \left(\cos 4x \cdot \frac{-\cos 5x}{5} - 4 \int -\sin 4x \, \frac{-\cos 5x}{5} \, dx \right)$$

$$= \frac{1}{5} \sin 4x \sin 5x + \frac{4}{25} \cos 4x \cos 5x + \frac{16}{25} \int \sin 4x \cos 5x \, dx$$

i.e. $$I = \frac{1}{5} \sin 4x \sin 5x + \frac{4}{25} \cos 4x \cos 5x + \frac{16}{25} I$$

Example

Evaluate

$$I = \int \sin 4x \cos 5x \, dx$$

So $\frac{9}{25} I = \frac{1}{5} \sin 4x \sin 5x + \frac{4}{25} \cos 4x \cos 5x + C$

$\Rightarrow \boxed{I = \frac{5}{9} \sin 4x \sin 5x + \frac{4}{9} \cos 4x \cos 5x + C}$