Integration of rational functions

E. Sköldberg
emil.skoldberg@nuigalway.ie
http://www.maths.nuigalway.ie/~emil/

School of Mathematics etc.
National University of Ireland, Galway

MA100

Partial fractions

A rational function is a function

$$f(x) = \frac{P(x)}{Q(x)} \quad \text{where } P(x), Q(x) \text{ are polynomials}$$

e.g. $\boxed{\dfrac{2x}{x^3+1}}$, $\boxed{\dfrac{x}{1-x}}$, $\boxed{\dfrac{x^3+17x-1}{x}}$

Some rational functions, we already know how to integrate:

$$\int \frac{1}{x^n}\,dx = \begin{cases} \dfrac{x^{-n+1}}{-n+1} + C = \dfrac{1}{1-n}\cdot\dfrac{1}{x^{n-1}} + C & n \geq 2 \\ \ln|x| + C & n = 1 \end{cases}$$

$$\int \frac{1}{x^2+1}\,dx = \tan^{-1}x + C$$

Example

Find

$$\int \frac{x^3 + x}{x - 1}\, dx$$

① Ensure that the degree of the numerator is less than the degree of the denominator by using long division.

$$
\begin{array}{r|l}
 & x^2 + x + 2 \\
\hline
x - 1 & x^3 + 0\cdot x^2 + x + 0 \\
 & -\;x^3 - x^2 \\
\hline
 & \quad x^2 + x \\
 & -\;x^2 - x \\
\hline
 & \qquad 2x \\
 & -\;2x - 2 \\
\hline
 & \qquad\quad 2
\end{array}
$$

Thus:

$$\frac{x^3+x}{x-1} = x^2+x+2 + \frac{2}{x-1}$$

Example

Find

$$\int \frac{x^3 + x}{x - 1}\, dx$$

So far: $\frac{x^3+x}{x-1} = x^2 + x + 2 + \frac{2}{x-1}$ $\underline{\text{so:}}$

$$\int \frac{x^3+x}{x-1}\,dx = \int \left(x^2 + x + 2 + \frac{2}{x-1}\right) dx =$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\log|x-1| + C$$

Factorization of the denominator

The situation is still that we want to integrate

$\int \frac{P(x)}{Q(x)}\,dx$. We will now assume that

$\deg P(x) < \deg Q(x)$. The next step is to

factorize the denominator $Q(x)$.

Ex $\frac{2x+3}{x^3-3x^2+2x}$. $x^3-3x^2+2x = x(x^2-3x+2)$

$= x(x-1)(x-2)$

so $\frac{2x+3}{x^3-3x^2+2x} = \frac{2x+3}{x(x-1)(x-2)}$

Factorization of the denominator

After fully factorising $Q(x)$, we have written

$Q(x)$ as a product of factors each

of the type:

- $x+b$ linear factor
- x^2+ax+b irreducible quadratic
- c constant

e.g. x^2+1

happens when

$\frac{a^2}{4}-b < 0$

The partial fraction theorem

Let $\dfrac{P(x)}{Q(x)}$ be a rational function with $\deg P(x) < \deg Q(x)$.

Then $\dfrac{P(x)}{Q(x)}$ can be written as a sum of terms each of the form

* $\dfrac{A}{(x+b)^j}$ if $(x+b)^n$ is a factor of $Q(x)$, $j \le n$
* $\dfrac{Ax+B}{(x^2+ax+b)^j}$ if $(x^2+ax+b)^n$ is a factor of $Q(x)$ and $j \le n$

The denominator is a product of distinct linear factors

Find

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

① The degree of the numerator is less than the degree of the denominator, so we do not need to divide

② Factorise the denominator:

$$2x^3 + 3x^2 - 2x = x(2x^2 + 3x - 2) =$$

$$= 2 \cdot x \cdot \left(x^2 + \tfrac{3}{2}x - 1\right) = 2 \cdot x \cdot (x+2)\left(x - \tfrac{1}{2}\right)$$

So: $$\frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x - \frac{1}{2}}$$ [By previous thm.]

The denominator is a product of distinct linear factors

Find
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-\frac{1}{2}}$$

$$= \frac{A\cdot 2\cdot(x+2)(x-\frac{1}{2})}{2\cdot x\cdot(x+2)(x-\frac{1}{2})} + \frac{B\cdot 2\cdot x(x-\frac{1}{2})}{2\cdot x\,(x+2)(x-\frac{1}{2})} + \frac{C\cdot 2\cdot x\cdot(x+2)}{2\cdot x(x+2)(x-\frac{1}{2})}$$

$$\Longrightarrow\quad x^2+2x-1 = 2A(x+2)(x-\tfrac{1}{2}) + 2Bx(x-\tfrac{1}{2}) + 2Cx(x+2)$$

Identifying the coefficients of $x^2, x, 1$ in LHS & RHS gives

$$\begin{cases} 1 = 2A + 2B + 2C \\ 2 = 3A - B + 4C \\ -1 = -2A \end{cases}$$

so $A = \frac{1}{2}$ $\Rightarrow$ $\begin{cases} 2B + 2C = 0 \\ -B + 4C = \frac{1}{2} \end{cases}$

The denominator is a product of distinct linear factors

Find

$$\int \frac{x^2 + 2x - 1}{2x^3 + 3x^2 - 2x}\, dx$$

The integrand is $\dfrac{A}{x} + \dfrac{B}{x+2} + \dfrac{C}{x-\frac{1}{2}}$

$A = \frac{1}{2}$ $\quad C = -B \quad -B + 4C = \frac{1}{2}$

$-B - 4B = \frac{1}{2}$

$-5B = \frac{1}{2} \quad B = -\frac{1}{10} \quad C = \frac{1}{10}$

so $\displaystyle\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\, dx = \int \frac{1}{2x} - \frac{1}{10(x+2)} + \frac{1}{10(x-\frac{1}{2})}\, dx$

$\displaystyle = \int \frac{1}{2x} - \frac{1}{10x+20} + \frac{1}{10x-5}\, dx = \frac{1}{2}\ln|2x| - \frac{1}{10}\ln|10x+20| + \frac{1}{10}\ln|10x-5| + C$

The denominator is a product of (possibly repeated) linear factors

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

① Make sure deg of numerator < deg of denominator

$$\begin{array}{r|l} & x + 1 \\ \hline x^3 - x^2 - x + 1 & x^4 \quad - 2x^2 + 4x + 1 \\ & -\left(x^4 - x^3 - x^2 + x\right) \\ \hline & x^3 - x^2 + 3x + 1 \\ & x^3 - x^2 - x + 1 \\ \hline & 4x \end{array}$$

So $\dfrac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = x + 1 + \dfrac{4x}{x^3 - x^2 - x + 1}$

The denominator is a product of (possibly repeated) linear factors

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

(2) Factorise the denominator:

Let $Q(x) = x^3 - x^2 - x + 1$, then $Q(1) = 0$

so $x - 1$ divides $Q(x)$

$$\begin{array}{r|l} & x^2 \quad\quad - 1 \\ \hline x - 1 & x^3 - x^2 - x + 1 \\ & x^3 - x^2 \\ \hline & \quad\quad -x + 1 \\ & \quad\quad -x + 1 \\ \hline & \quad\quad\quad 0 \end{array}$$

so $Q(x) = (x-1)(x^2-1)$

$= (x-1)\underbrace{(x+1)(x-1)}_{x^2-1}$

$= (x-1)^2(x+1)$

The denominator is a product of (possibly repeated) linear factors

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

(3) Decompose into partial fractions:

$$\frac{P(x)}{Q(x)} = x+1 + \frac{4x}{x^3-x^2-x+1} = x+1 + \underbrace{\frac{4x}{(x-1)^2(x+1)}}$$

$$\frac{4x}{(x-1)^2(x+1)} = \frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{C}{x+1}$$

Multiply by $Q(x)$:

$$4x = A(x+1) + B\overbrace{(x+1)(x-1)}^{x^2-1} + C\overbrace{(x-1)^2}^{x^2-2x+1}$$

$$\begin{cases} 0 = B + C \\ 4 = A - 2C \\ 0 = A - B + C \end{cases} \qquad \begin{cases} A = 2 \\ B = 1 \\ C = -1 \end{cases}$$

The denominator is a product of (possibly repeated) linear factors

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

So far:

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1} = \int x + 1 + \frac{4x}{x^3 - x^2 - x + 1}\, dx$$

$$= \int x + 1 + \overbrace{\frac{2}{(x-1)^2}}^{2(x-1)^{-2}} + \frac{1}{x-1} - \frac{1}{x+1}\, dx =$$

$$= \frac{x^2}{2} + x - \underbrace{\frac{2}{x-1}}_{2(x-1)^{-1}} + \ln|x-1| - \ln|x+1| + C$$

The denominator is a product of (possibly repeated) linear factors

Find

$$\int \frac{x^4 - 2x^2 + 4x + 1}{x^3 - x^2 - x + 1}\, dx$$

The denominator contains irreducible quadratic factors

Find

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx$$

① $P(x) = 2x^2 - x + 4 \qquad Q(x) = x^3 + 4x$

$\deg P < \deg Q$; no division necessary!

② Factorise the denominator: $Q(x) = x^3 + 4x = x\underbrace{(x^2+4)}_{\text{irreducible}}$

③ Decompose using partial fractions:

$$\frac{2x^2 - x + 4}{x^3 + 4x} = \frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Cx + D}{x^2 + 4}$$

The denominator contains irreducible quadratic factors

Find

$$\int \frac{2x^2 - x + 4}{x^3 + 4x}\, dx$$

$$\frac{2x^2 - x + 4}{x(x^2+4)} = \frac{A}{x} + \frac{Bx + C}{x^2+4}$$

gives the system:

Multiply both sides by $x(x^2+4)$

$$2x^2 - x + 4 = A(x^2+4) + (Bx+C)x$$

$$2x^2 - x + 4 = Ax^2 + 4A + Bx^2 + Cx$$

$$\Rightarrow \begin{cases} 2 = A + B \\ -1 = C \\ 4 = 4A \end{cases} \Rightarrow \begin{cases} A = 1 \\ B = 1 \\ C = -1 \end{cases}$$

Cont.

So the integral is equal to

$$\int \frac{1}{x} + \frac{x-1}{x^2+4}\,dx = \int \frac{1}{x} + \frac{x}{x^2+4} - \frac{1}{x^2+4}\,dx$$

let us integrate each term separately:

- $\int \frac{1}{x}\,dx = \ln|x| + c$
- $\int \frac{x}{x^2+4}\,dx = \left[\begin{array}{l} t = x^2+4 \quad dt = 2x\,dx \\ \frac{dt}{dx} = 2x \end{array}\right] \int \frac{1}{2}\cdot\frac{1}{t}\,dt$

  $= \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln\left(x^2+4\right)$
- $\int \frac{1}{x^2+4}\,dx = \frac{1}{4}\int \frac{1}{\frac{x^2}{4}+1}\,dx = \frac{1}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx = \left[\begin{array}{l} t = \frac{x}{2} \\ dt = \frac{1}{2}\,dx \end{array}\right]$

Cont.

$$= \frac{1}{2} \int \frac{1}{t^2+1}\, dt = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

Summing all the integrals of the terms:

$$\int \frac{2x^2 - x + 4}{x^3 + 4x} = \ln|x| + \frac{1}{2}\ln(x^2+4) - \frac{1}{2}\tan^{-1}\frac{x}{2} + c$$

The denominator contains repeated irreducible quadratic factors

Find

$$\int \frac{1 - x + 2x^2 - x^3}{x(x^2+1)^2}\,dx$$

Here we would get a partial fraction expansion

$$\boxed{\frac{A}{x} + \frac{Bx + C}{(x^2+1)^2} + \frac{Dx + E}{x^2+1}}$$