Strategies of integration

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MA100

A four-step plan

1. Simplify the integrand.
2. Look for an obvious substitution.
3. Classify the integrand accoring to its form
    1. Trigonometric functions.
    2. Rational functions.
    3. Integration by parts.
4. Try again!

Simplify the integrand

$$\int \sqrt{x}(1+\sqrt{x})\,dx$$

$$\int \sqrt{x}\,(1+\sqrt{x})\,dx = \int (\sqrt{x} + x)\,dx = \int (x^{1/2} + x)\,dx$$

$$= \frac{x^{3/2}}{3/2} + \frac{x^2}{2} + C = \frac{2}{3}x^{3/2} + \frac{1}{2}x^2 + C$$

$$\int (\sin x + \cos x)^2\,dx$$

$$\int (\sin x + \cos x)^2\,dx = \int \left(\sin^2 x + 2\sin x\cos x + \cos^2 x\right)dx =$$

$$= \int 1 + \underbrace{2\sin x\cos x}_{\sin 2x}\,dx = \int 1 + \sin 2x\,dx = x - \frac{1}{2}\cos 2x + C$$

Look for an obvious substitution

$$\int \frac{x^2}{x^3 - 1}\,dx$$

$$\int \frac{x^2}{x^3-1}\,dx = \left[\begin{array}{ll} u = x^3 - 1 & du = 3x^2\,dx \\ \frac{du}{dx} = 3x^2 & \end{array}\right]$$

$$\int \frac{1}{u}\,\frac{1}{3}\,\underbrace{3x^2\,dx}_{du} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$$

$$= \frac{1}{3}\ln|x^3 - 1| + C$$

Classify the integrand according to its form

Trigonometric functions:

$\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x + \cos^2 x = 1$

$\cos^2 x = 1 - \sin^2 x$

$$\int \sin^2 x \cos^7 x \, dx$$

$$\int \sin^2 x \cos^7 x \, dx = \int \sin^2 x \cdot \cos^2 x \cdot \cos^2 x \cdot \cos^2 x \cdot \cos x \, dx$$

$$= \int \sin^2 x \left(1 - \sin^2 x\right)^3 \cos x \, dx = \left[u = \sin x, \quad \frac{du}{dx} = \cos x, \quad \cos x \, dx = du \right]$$

$$= \int u^2 (1-u^2)^3 \, du = \int u^2 (1 - 3u^2 + 3u^4 - u^6) \, du =$$

$$= \int u^2 - 3u^4 + 3u^6 - u^8 \, du$$

$$= \frac{u^3}{3} - \frac{3u^5}{5} + \frac{3u^7}{7} - \frac{u^9}{9} + C$$

$$= \frac{\sin^3 x}{3} - \frac{3\sin^5 x}{5} + \frac{3\sin^7 x}{7} - \frac{\sin^9 x}{9} + C$$

$$\int \cos^4 x \, dx = \int \left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \cos^2 2x \, dx =$$

$$= \frac{1}{4} \int 1 + 2\cos 2x + \frac{1+\cos 4x}{2} \, dx =$$

$$= \frac{1}{4} \left(x + \sin 2x + \frac{x}{2} + \frac{\sin 4x}{8} \right) + C$$

Classify the integrand according to its form

Rational functions:

$$\int \frac{x^2 - 2x + 3}{x^2(x^2+1)}$$

$\int \frac{x^2-2x+3}{x^2(x^2+1)}\,dx$. The method of partial fractions

gives: $\frac{x^2-2x+3}{x^2(x^2+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{Cx+D}{x^2+1}$

Solve for A, B, C and integrate.

Integration by parts

$$\int x^2 e^{2x}\, dx$$

$$\int x^2 \cdot e^{2x}\, dx = x^2 \cdot \frac{e^{2x}}{2} - \int 2x \cdot \frac{e^{2x}}{2}\, dx =$$

$$= \frac{1}{2} x^2 e^{2x} - \int x \cdot e^{2x}\, dx = \frac{1}{2} x^2 e^{2x} - \left(x \frac{e^{2x}}{2} - \int 1 \cdot \frac{e^{2x}}{2}\, dx \right)$$

$$= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} - \int \frac{1}{2} e^{2x}\, dx =$$

$$= \frac{1}{2} x^2 e^{2x} - \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + C$$

Elementary functions

The elementary functions are

- Polynomials x^n. Actually all powers of x x^b
- Exponentials a^x.
- Logarithms.
- Trigonometric functions $\sin x$, $\cos x$ etc.
- Inverse trigonometric functions, $\sin^{-1} x$, $\cos^{-1} x$ etc.

And every function that can be built from these by addition, subtraction, multiplication, division and composition.

Ex: $\dfrac{\sin x^2}{e^{\sqrt{\cos x}}}$ and $\ln\left(\dfrac{\tan^{-1} x}{x}\right)$ are elementary

Can we integrate every continuous function?

$$\int e^{x^2}\,dx \qquad \int \frac{1}{\ln x}\,dx \qquad \int \frac{\sin x}{x}\,dx$$

- Every continuous function is integrable
  i.e has an antiderivative

But

- The antiderivative of most functions is not elementary (The three functions above are examples of this)