# Differential Equations

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# Population growth

Suppose we have a population of rabbits

We have 2 variables

- $t$ for time (independent var)
- $P$ the population (# of rabbits)
  (dependent var.)

One reasonable hypothesis:

the growth of $P$ is proportional to $P$ itself

(we have unlimited availability of food, space etc)

A mathematical model: $\boxed{\dfrac{dP}{dt} = k \cdot P}$

# Population growth (cont.)

$\frac{dP}{dt} = kP$ . Consequences of the model:

$P(t) > 0$ positive population (if at any time $P > 0$)

$P'(t) > 0$ so $P(t)$ is always increasing

Solving the Differential equation:

$P(t) = C \cdot e^{kt}$

$\left| \frac{d}{dt}\left(C \cdot e^{kt}\right) = k \cdot C e^{kt} \right.$

So this function is a solution for all $C$

This is an infinite family of solutions

$C = C \cdot e^{0} = P(0)$ ; the initial population.

# The logistic equation

A slightly more realistic model of population growth would have a carrying capacity, $K$, the population size that the environment would sustain.

Assumptions: $\dfrac{dP}{dt} \simeq kP$ if $P$ is small compared to $K$

$$\frac{dP}{dt} < 0 \quad \text{if } P > K.$$

An example of an equation satisfying these assumptions is the logistic equation:

$$\boxed{\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{K}\right)}$$

# The logistic equation (cont.)

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

Two very specific solutions are

Two equilibria

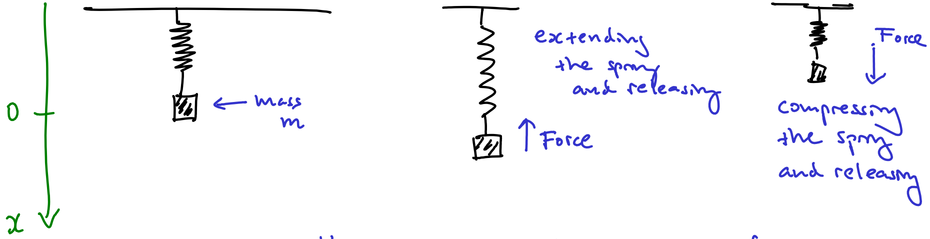
$P(t) = 0$ :   L.H.S. : $\frac{dP}{dt} = 0$

R.H.S. : $k \cdot P \cdot \left(1 - \frac{P}{K}\right) = k \cdot 0 \cdot 1 = 0$

$P(t) = K$ :   L.H.S. $\frac{dP}{dt} = 0$

R.H.S. : $k \cdot P \cdot \left(1 - \frac{P}{K}\right) = k \cdot K \cdot 0 = 0$

# The motion of a spring




According to Hooke's law, the restoring force is equal to $-kx$, $k$ a constant dep on the spring

Newton tells us that $F = m \cdot a$ so we get

$$ma = -kx$$

$$m\frac{d^2x}{dt^2} = -kx \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}x$$

# The motion of a spring (cont.)

$$\frac{d^2x}{dt^2} = -\frac{k}{m}x \qquad (*)$$

Special case: $\frac{k}{m} = 1$, so $\frac{d^2x}{dt^2} = -x$

one solution is $x(t) = \sin t$

because $\frac{dx}{dt} = \cos t$ $\quad \frac{d^2x}{dt^2} = -\sin t = -x$

Back to the general case:

one solution to $(*)$ $\quad x(t) = \sin\left(\sqrt{\frac{k}{m}}\,t\right)$

$$\frac{dx}{dt} = \sqrt{\frac{k}{m}} \cdot \cos\left(\sqrt{\frac{k}{m}}\,t\right) \qquad \frac{d^2x}{dt^2} = -\frac{k}{m}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$= -\frac{k}{m}x.$$

General sol:

$$\boxed{x(t) = A\sin\left(\sqrt{\frac{k}{m}}\,t\right) + B\cos\left(\sqrt{\frac{k}{m}}\,t\right)}$$