Differential equation

- A differential equation is an equation containing an unknown function and one or more of its derivatives.
- The order of a differential equation is the order of the highest derivative of the unknown function that occurs in the equation.

Example

Show that the function $y = \frac{1}{x+C}$ is a solution to the differential equation $y' = -y^2$ for all C.

L.H.S. : $y' = \frac{d}{dx}\left(\frac{1}{x+c}\right) = \frac{d}{dx}\left((x+C)^{-1}\right) =$

$= -1 \cdot (x+C)^{-2} = -\frac{1}{(x+c)^2} = -\left(\frac{1}{x+C}\right)^2$

$= -y^2$

Example (cont.)

Solve the initial value problem

$$y' = -y^2, \quad y(0) = 0.5.$$

From previous slide, we know that $y = \dfrac{1}{x+C}$ is a solution to the DE for all C.

Using that $y(0) = \frac{1}{2}$, we get:

$$\frac{1}{0+C} = \frac{1}{2} \implies C = 2$$

So the solution is $y(t) = \dfrac{1}{x+2}$.

Separable equations

Definition

A separable differential equation is a first-order differential equation of the form

$$\frac{dy}{dx} = g(x)f(y)$$

Examples of separable equations

- $y' = \sin x \cdot \cos y$
- $\dfrac{dy}{dx} = \underbrace{xy + x + y + 1}_{(x+1)(y+1)}$

since

The solution of a separable equation

Suppose we are given the separable D.E.

$$\frac{dy}{dx} = g(x)f(y)$$

Rewrite this as $\frac{1}{f(y)}\,dy = g(x)\,dx$

and integrate $\underbrace{\int \frac{1}{f(y)}\,dy}_{\text{function in } y} = \underbrace{\int g(x)\,dx}_{\text{function in } x}$

and then solve for y

Example

Find the solution of

$$\frac{dy}{dx} = \frac{x^2}{y^2}$$

This is a separable equation since $\frac{dy}{dx} = x^2 \cdot \frac{1}{y^2}$

and we write

$$y^2\,dy = x^2\,dx$$

then integrate:

$$\int y^2\,dy = \int x^2\,dx$$

$$\frac{y^3}{3} = \frac{x^3}{3} + C$$

$$y^3 = x^3 + 3C$$

$$y = \sqrt[3]{x^3 + 3C}$$

Optional:

We can tidy up by letting $C_1 = 3C$ and get

$$y = \sqrt[3]{x^3 + C_1}$$

Example

Find the solution of

$$y' = x^2 y$$

Write it in Leibniz form $\left(\frac{dy}{dx} \text{ instead of } y'\right)$,

$$\frac{dy}{dx} = x^2 \cdot y$$

$$\frac{1}{y}\,dy = x^2\,dx$$

Now integrate both sides.

Don't forget to include a constant of integration

$$\int \frac{1}{y}\,dy = \int x^2\,dx$$

$$\ln|y| = \frac{x^3}{3} + C$$

Solve for y:

$$e^{\ln|y|} = e^{\frac{x^3}{3}+C}$$

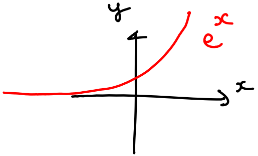
$$|y| = e^{\frac{x^3}{3}+C} = e^C \cdot e^{x^3/3}$$

Example

$|y| = a$
$\Rightarrow y = \pm a$

Find the solution of

$$y' = x^2 y$$



So far : $|y| = e^c \cdot e^{x^3/3}$

Thus $y = \underbrace{\pm e^c}_{\text{any number} \neq 0} e^{x^3/3}$

[Note : $y(x) = 0$ is a solution, since $y' = 0$, $x^2 \cdot y = x^2 \cdot 0 = 0$]

$\Rightarrow$ $y = D \cdot e^{x^3/3}$, D is an arbitrary constant

Let us check that we have a solution :

$LHS = y' = D \cdot e^{x^3/3} \cdot \frac{3x^2}{3} = x^2 \cdot D e^{x^3/3} = x^2 \cdot y$

Example

Find the solution of

$$y' = x^2 y$$

ℓ not 1

A tank contains 1000l of beer with 4% alcohol. Beer with 6% alcohol is pumped into the tank at a rate of 20l per minute, and the mixture is pumped out at the same rate. What is the percentage of alcohol in the mixture after 1 hour?

Let $y(t)$ be the volume of alcohol in tank at time t. thus $y(0) = 0.04 \cdot 1000 = 40$

The rate of change of y is given by

$$\frac{dy}{dt} = (\text{inflow of alcohol /min}) - (\text{outflow of alc /min})$$

$$= 0.06 \cdot 20 - \frac{y(t)}{1000} \cdot 20 = 1.2 - 0.02y$$

Summing up $\quad \frac{dy}{dt} = 1.2 - 0.02y \quad . \quad y(0) = 40$

A tank contains 1000l of beer with 4% alcohol. Beer with 6% alcohol is pumped into the tank at a rate of 20l per minute, and the mixture is pumped out at the same rate. What is the percentage of alcohol in the mixture after 1 hour?

$$\frac{dy}{dt} = 1.2 - 0.02y \quad , \quad y(0) = 40$$

This is a separable equation:

$$\frac{1}{1.2 - 0.02y}\, dy = 1 \cdot dt$$

integrate

$$\int \frac{1}{1.2 - 0.02y}\, dy = \int 1\, dt$$

$$-50 = \frac{1}{-0.02}$$

$$-50 \cdot \ln|1.2 - 0.02y| = t + C$$

$$\ln|1.2 - 0.02y| = -\frac{t+C}{50}$$

$$1.2 - 0.02y = \pm e^{-\frac{t+C}{50}}$$

$$0.02y = 1.2 \pm e^{-\frac{t+C}{50}}$$

$$y = 60 \pm 50 e^{-\frac{t+C}{50}}$$

T. B. C.