Models for population growth

E. Sköldberg
emil.skoldberg@nuigalway.ie
http://www.maths.nuigalway.ie/~emil/

School of Mathematics etc.
National University of Ireland, Galway

MA100

Example continued from last week

A tank contains 1000l of beer with 4% alcohol. Beer with 6% alcohol is pumped into the tank at a rate of 20l per minute, and the mixture is pumped out at the same rate. What is the percentage of alcohol in the mixture after 1 hour?

$$-50 \ln |1.2 - 0.02y| = t + C$$

We use that $y(0) = 40$ to calculate C:

$$-50 \ln |1.2 - 0.02 \cdot 40| = 0 + C$$

$$-50 \ln 0.4 = C$$

So our equation becomes $-50 \ln |1.2 - 0.02y| = t - 50 \ln 0.4$

$$\Rightarrow \ln |1.2 - 0.02y| = \ln 0.4 - \frac{t}{50}$$

Example continued from last week

A tank contains 1000l of beer with 4% alcohol. Beer with 6% alcohol is pumped into the tank at a rate of 20l per minute, and the mixture is pumped out at the same rate. What is the percentage of alcohol in the mixture after 1 hour?

$$\ln|1.2-0.02y| = \ln 0.4 - \frac{t}{50}$$

exponentiate both sides to get rid of ln

$$|1.2-0.02y| = e^{\ln 0.4 - \frac{t}{50}}$$

$$|\underbrace{1.2-0.02y}_{\geqslant 0 \text{ at } t=0}| = e^{\ln 0.4} \cdot e^{-t/50} = \underbrace{0.4\, e^{-\frac{t}{50}}}_{\geqslant 0}$$

$$\Rightarrow \quad 1.2-0.02y = 0.4\, e^{-t/50} \Rightarrow \boxed{y = 60-20\, e^{-t/50}}$$

Ans:

Alc. after 1 hr $= y(60) = 60-20e^{-1.2}$

$$\Rightarrow \frac{60-20e^{-1.2}}{1000}$$

The law of natural growth

P is the population (depends of t)

t is the time

$$\frac{dP}{dt} = kP$$

$\frac{dP}{dt}$ is the rate of change of the population

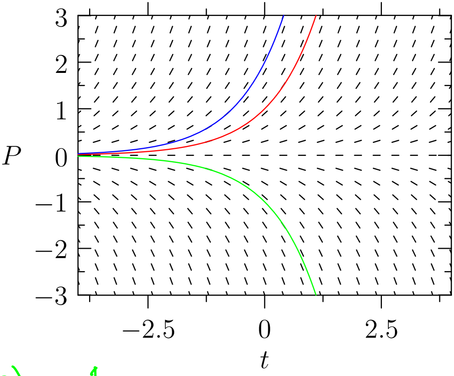
Blue curve:

solution passing through (0, 2)

i.e P(0) = 2

Red curve:

solution s.t P(0) = 1



<u>Ex</u>

<u>k = 1</u>

$\frac{dP}{dt} = P$

Green curve:

Solution s.t. P(0) = −1

The law of natural growth

$$\frac{dP}{dt} = kP$$

This differential equation is separable.

$$\left[\frac{dP}{dt} = f(P) \cdot g(t) \right]$$

So $\frac{dP}{dt} = k \cdot P$ Collect P in LHS, t in RHS

$\Rightarrow \frac{1}{P} \cdot dP = k \cdot dt$ Now: integrate

$$\int \frac{1}{P}\, dP = \int k\, dt$$

$\ln|P| = k \cdot t + C$ Exponentiate

$|P| = e^{\ln|P|} = e^{k \cdot t + C} = e^{C} \cdot e^{kt}$

So $P = \pm e^{C} \cdot e^{k \cdot t}$

or $\boxed{P = A \cdot e^{kt}}$

for some A

The law of natural growth

Theorem

The solution to the initial value problem

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

is

$$P(t) = P_0 e^{kt}$$

From the last slide, we know that

$P(t) = A e^{k \cdot t}$ Set $t = 0$, so $P(0) = P_0$

$P_0 = A \cdot e^{k \cdot 0} = A e^0 = A$.

Hence, we get solution $P(t) = P_0 e^{kt}$.

Radioactive decay

Radioactive material decays at a rate that is proportional to the amount of the material. This gives the differential equation

$$\frac{dm}{dt} = km$$

where m is the mass of the sample. The rate of decay is usually measured by the *half-life*, the time it takes for half of a sample to decay.

Radioactive decay (cont.)

Cesium–137 has a half live of 30.17 years. A sample of cesium has a mass of 1000g. How much remains after 10 years?

The differential equation is $\frac{dm}{dt} = km$

where t is the time in years

m is the mass of cesium in the sample.

The solution is $m(t) = m_0 e^{k \cdot t}$ where $m_0 = 1000$

so $m(t) = 1000 e^{k \cdot t}$

We want to find $m(10)$

We know that $m(30.17) = 500$

so $500 = 1000 e^{30.17 k}$

$e^{30.17 k} = \frac{1}{2}$

$30.17 k = \ln \frac{1}{2} = -\ln 2$

$k = -\frac{\ln 2}{30.17}$

Radioactive decay (cont.)

Cesium–137 has a half live of 30.17 years. A sample of cesium has a mass of 1000g. How much remains after 10 years?

$m(t) = 1000 e^{-\frac{\ln 2}{30.17} t}$ is the solution, so

$m(10)$, the amount of cesium after 10 years is

$$m(10) = 1000 e^{-\frac{\ln 2}{30.17} 10} \approx 795 g$$