Antiderivatives

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Antiderivatives

Sometimes we can measure the derivative of a function, without an easy way of measuring the function itself. Examples might be

- We can measure the speed of a car, and want to find the distance travelled after a given time.
- We can measure the rate at which water flows out of a reservoir, and want to find the amount that has flowed out after a month.

In each of these cases, we want to find a function $F(x)$ whose derivative is a given function $f(x)$, that is, we want to the opposite of differentiation.

Stewart: Section 4.9

Definition

A function F is an *antiderivative* of the function f on the interval I if $F'(x) = f(x)$ for all $x \in I$.

Examples

Let $F(x) = x^3$, then $F'(x) = 3x^2$

So x^3 is an antiderivative of $3x^2$ on $\mathbb{R}$

Let $G(x) = \sin x^2$, then $g(x) = G'(x) = 2x \cos x^2$

so $\sin x^2$ is the antiderivative of $2x \cos x^2$.

We might have many antiderivatives of a function

e.g $f(x) = x+2$ $g(x) = x-13$, then

$f'(x) = g'(x) = 1$. So $x+2$ and $x-13$ are two different antiderivatives of 1

Theorem

If F is an antiderivative of f on I, then the most general antiderivative of f is $F(x) + C$ where C is a constant.

Why? Suppose $F_1(x)$ and $F_2(x)$ are both antiderivatives of $f(x)$. Thus $F_1'(x) = f(x) = F_2'(x)$

and then $\frac{d}{dx}\left(F_1(x) - F_2(x)\right) = f(x) - f(x) = 0$

By the mean value theorem $F_1(x) - F_2(x) = C$

for C a constant.

Examples

Find the general antiderivative of the functions $f(x) = \cos x$, $g(x) = e^x$ and $h(x) = x^2$.

① Let $F(x) = \sin x$, then $F'(x) = \cos x = f(x)$

so the general antiderivative of $f(x)$ is $\sin x + C$

for a constant C.

② Let $G(x) = e^x$, then $G'(x) = e^x = g(x)$

so the general antiderivative of $g(x)$ is $e^x + C$

③ Let $H(x) = \frac{1}{3}x^3$, then $H'(x) = \frac{1}{3} \cdot 3x^2 = x^2 = h(x)$

so the general antiderivative of $h(x)$ is $\frac{x^3}{3} + C$

Table of antidifferentiation formulas

Suppose that f, g, F, G are functions such that $F'(x) = f(x)$ and $G'(x) = g(x)$. Then we have

Function	Particular antiderivative
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
$x^n, n \neq -1$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Find the most general antiderivative of the following functions

- $f(x) = \frac{11}{x^{10}}$

We can write $f(x) = \frac{11}{x^{10}} = 11 \cdot \frac{1}{x^{10}} = 11 \cdot x^{-10}$

So, let $F(x) = 11 \cdot \frac{x^{-10+1}}{-10+1} = 11 \cdot \frac{x^{-9}}{-9} = -\frac{11}{9x^9}$

So the most general antiderivative of $f(x)$ is $\boxed{-\frac{11}{9x^9} + C}$

- $g(x) = \sin x + \cos 2x$

We have a sum of functions, so we can find antiderivatives of $\sin x$ and $\cos 2x$ separately.

$\sin x$ has an antiderivative $-\cos x$

$\cos 2x$ —— " —— $\frac{1}{2} \sin 2x$

Therefore, the most general antiderivative of $\sin x + \cos 2x$ is $-\cos x + \frac{1}{2}\sin 2x + C$

Example

Find $f(x)$ if $f''(x) = 60x^4 - 2e^x$.

$f'(x)$ is an antiderivative of $60x^4 - 2e^x$

so $f'(x) = 60 \cdot \frac{x^5}{5} - 2e^x + C = 12x^5 - 2e^x + C$

Thus $f(x)$, which is an antiderivative of $f'(x)$ is

$f(x) = 12 \cdot \frac{x^6}{6} - 2e^x + Cx + D$

$= 2x^6 - 2e^x + Cx + D$

Rectilinear motion

Rectilinear motion or *linear motion* is motion along a straight line. Supposing that the position of a particle at the time t is given by the function $s(t)$, the velocity of the particle is given by $v(t)$ and its acceleration is given by $a(t)$. In that case we have the relationships:

- $v(t) = s'(t)$
- $a(t) = v'(t)$

We thus have a need for antiderivatives if we know the velocity or acceleration and want to find the position.

Rectilinear motion

A car is travelling with the speed of 72 $\mathrm{km/h}$ when it brakes with a constant deceleration of 5 $\mathrm{m/s^s}$. What distance has the car travelled when it comes to a halt?

72 km/h is 20 m/s.

Now we know that $a(t) = -5$, so

$v(t) = -5t + C$ and $s(t) = -\frac{5}{2}t^2 + Ct + D$

We know that $v(0) = 20$, so $C = 20$

We can set $s(0) = 0$, so $0 = 0 + 20 \cdot 0 + D$

So $D = 0$.

Now $s(t) = 20t - \frac{5}{2}t^2$

We have stopped after four seconds and

$s(4) = 20 \cdot 4 - \frac{5}{2}4^2 = 80 - 40 = 40$

Thus, the car has travelled 40 m