The logistic model for population growth

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The logistic model

The *logistic differential equation* is given by

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

where K is the *carrying capacity*.

$P(t)$ · the unknown function, depending on $\underline{t}$
giving the size of population at time t.

k : constant of proportionality.
k large for bacteria (pop. grows quickly)
k small for blue whales (pop grows slowly)

K : the size of pop that the environment
can long term sustain.

The logistic model

The *logistic differential equation* is given by

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$$

$k > 0$

where K is the *carrying capacity*.

What can we say about the behaviour of $P(t)$ without solving the D.E.

- If $P(t) > 0$ but $P(t) << K$, then $1 - \frac{P}{K} \approx 1$ and we have exponential growth.
- If $0 < P(t) < K$, then $1 - \frac{P}{K} > 0$, so $\frac{dP}{dt} > 0$ so $P(t)$ is increasing
- If $P(t) > K$, then $1 - \frac{P}{K} < 0$, so $\frac{dP}{dt} < 0$ and $P(t)$ is decreasing
- If $P(t) = K$, then $1 - \frac{P}{K} = 0$, so $\frac{dP}{dt} = 0$, and P is constant

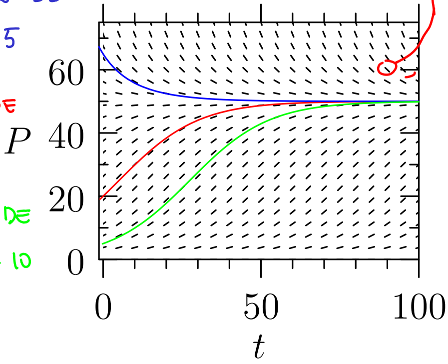
The logistic model – Example

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right), \qquad k = 0.08, K = 50$$

Blue : Solution to DE satisfying $P(0) = 65$

Red : Solution to DE satisfying $P(0) = 20$

Green : Solution to DE satisfying $P(0) = 10$



line with slope $0.08 \cdot 60 \cdot \left(1 - \frac{60}{50}\right)$

$= -0.96$

The solution of the logistic differential equation

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{K}\right)$$

P is a function of t
k is a constant
K " "

This is a separable D.E.

$$\frac{dP}{dt} = \underbrace{k}_{\text{function of } t} \cdot \underbrace{P\left(1 - \frac{P}{K}\right)}_{\text{function of } P}$$

(well – a constant function)

$$\Rightarrow \frac{1}{P\left(1-\frac{P}{K}\right)} dP = k\, dt$$

Now, integrate!

$$\int \frac{1}{P(1-\frac{P}{K})} dP = \int k\, dt$$

- $\int k\, dt = kt + C$
- $\int \frac{1}{P(1-\frac{P}{K})} dP = \int \frac{K}{P(K-P)} dP = \int \frac{1}{P} + \frac{1}{K-P}\, dP =$

$$= \ln|P| - \ln|K-P| = \ln\left|\frac{P}{K-P}\right|$$

The solution of the logistic differential equation

Thus, so far:

$$\ln \left| \frac{P}{K-P} \right| = kt + C$$

$$\ln \left| \frac{K-P}{P} \right| = -kt - C \qquad \text{since } \ln \frac{1}{a} = -\ln a$$

Exponentiate:

$$\left| \frac{K-P}{P} \right| = e^{-kt-C} = e^{-C} \cdot e^{-kt}$$

$$\frac{K-P}{P} = \underbrace{\pm e^{-C}}_{\text{call this } A.} \cdot e^{-kt}$$

so $\boxed{\underbrace{\frac{K-P}{P}}_{\frac{K}{P}-1} = A e^{-kt}}$ for some A $\Rightarrow$ $\frac{K}{P} = 1 + A e^{-kt}$

$$\boxed{P = \frac{K}{1 + A e^{-kt}}}$$

The solution of the logistic differential equation

So the general solution to

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) \qquad P(0) = P_0$$

is

$$\boxed{P(t) = \frac{K}{1 + Ae^{-kt}} \qquad \text{for } A \in \mathbb{R}}$$

What is the value of A? $t = 0$ gives $P(0) = P_0$

(If we call the initial population P_0)

From last slide: $\frac{K-P}{P} = Ae^{-kt}$, so $\boxed{\frac{K-P_0}{P_0} = A}$

The solution of the logistic differential equation (cont.)

The solution of the logistic differential equation (cont.)

Theorem

The solution to the logistic differential equation

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right), \qquad P(0) = P_0$$

is

$$P(t) = \frac{K}{1 + Ae^{-kt}}, \qquad \text{where } A = \frac{K - P_0}{P_0}$$

Example

Solve the initial-value problem

$$\frac{dP}{dt} = 0.08\left(1 - \frac{P}{1000}\right)P \qquad P(0) = 100.$$

and hence find $P(40)$ and $P(80)$ as well as the time when the population reaches 900.

An observation: K, the carrying capacity, is 1000

The solution to the equation is

$$P(t) = \frac{K}{1 + Ae^{-kt}} \quad \text{where} \quad K = 1000, \quad k = 0.08$$

where $A = \dfrac{K - P_0}{P_0} = \dfrac{900}{100} = 9$

Thus: $P(t) = \dfrac{1000}{1 + 9e^{-0.08t}}$

Example (cont.)

$$P(t) = \frac{1000}{1+9e^{-0.08t}}$$

so $P(40) = \dfrac{1000}{1+9e^{-3.2}} \approx 732$

$P(80) = \dfrac{1000}{1+9e^{-6.4}} \approx 985$

Now, solve $P(t) = 900$.

$$\frac{1000}{1+9e^{-0.08t}} = 900 \Rightarrow 1+9e^{-0.08t} = 9$$

$$9e^{-0.08t} = 8$$

$$e^{-0.08t} = 8/9$$

$$-0.08t = \ln\frac{8}{9}$$

$$t = \frac{\ln 8/9}{-0.08} \approx ?$$