Revision

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MA100

Q5a(i) S11

Explain, in terms of areas in the plane, what is meant by the *definite integral*

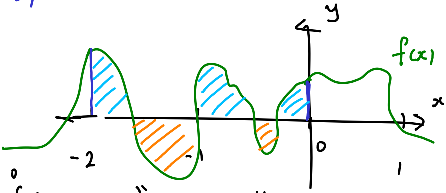
$$\int_{-2}^{0} f(x)\, dx$$

Draw a diagram to support your answer.

The integral $\int_{-2}^{0} f(x)\,dx$ is the signed area of the region bounded by

1) The graph of $f(x)$
2) The x-axis
3) The lines $y = -2$ and $y = 0$

The signed area means that regions above x-axis are counted as positive, regions below as negative



$\int_{-2}^{0} f(x)$ is "blue area" - "orange area"

Q5a(ii) S11

Explain what is meant by the *indefinite integral*

$$\int f(x)\,dx$$

Give an example to support your answer.

① A definite integral is a number.

② An indefinite integral is a function*

*) or a family of functions

The indefinite integral $\int f(x)\,dx$ is a function $F(x)$

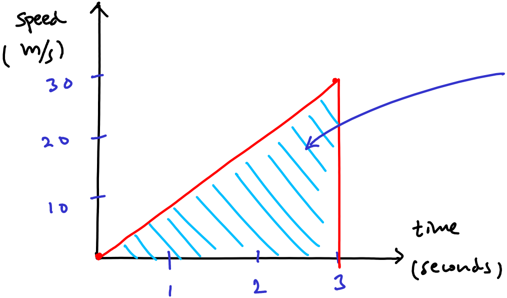
s.t $F'(x) = f(x)$. i.e. an antiderivative of $f(x)$.

Note: If $F'(x) = f(x)$, then $G'(x) = f(x)$ when $G(x) = F(x) + C$.

Ex $\int x^2\,dx = \frac{x^3}{3} + C$, where C is an arbitrary constant.

Q5b S11

A dart is dropped from the leaning tower in Pisa and accelerates steadily to a speed of 28 metres per second over a period of three seconds. Plot the graph of the dart's speed against time, and hence or otherwise calculate the distance travelled by the dart during this three-second period.



The distance travelled is the area of the shaded region

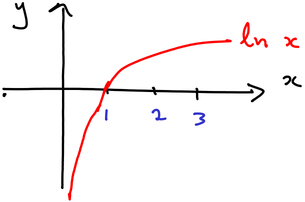
$$\left(\text{Since the area} = \int_0^3 \text{speed}(t)\, dt = \begin{matrix}\text{distance}\\ \text{travelled}\end{matrix} \right)$$

The area is $\dfrac{3 \cdot 28}{2} = 3 \cdot 14 = 42$

Answer: The dart has travelled 42 m.

Q5c S11

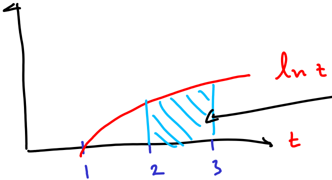
For $x \geq 1$, define $A(x)$ by



$$A(x) = \int_3^x \ln t \, dt$$

Draw a diagram indicating the geometric meaning of $A(2)$. What does the fundamental theorem of calculus tell us about the derivative $A'(x)$ of $A(x)$?

$A(2) = \int_3^2 \ln t \, dt = -\int_2^3 \ln t \, dt = -$(the area of the shaded region in the diagram)



Q5c S11

For $x \geq 1$, define $A(x)$ by

$$A(x) = \int_3^x \ln t \, dt$$

Draw a diagram indicating the geometric meaning of $A(2)$. What does the fundamental theorem of calculus tell us about the derivative $A'(x)$ of $A(x)$?

The fundamental theorem of calculus says that

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x) \quad \left(\begin{array}{l} f(x) \text{ is assumed to} \\ \text{be continuous} \end{array} \right)$$

Thus $\quad A'(x) = \dfrac{d}{dx} \displaystyle\int_3^x \ln t \, dt = \ln x.$

Q6a S11

Determine the following integral:

$$\int_0^3 e^{3-x}\,dx$$

Let us solve this by a change of variables:

$t = 3 - x$

$\frac{dt}{dx} = -1$

$dt = -dx$

$x = 0 \Rightarrow t = 3$

$x = 3 \Rightarrow t = 0$

$$\int_0^3 e^{3-x}\,dx = -\int_3^0 e^t\,dt = \int_0^3 e^t\,dt =$$

$$= \left[e^t\right]_0^3 = e^3 - e^0 = \underline{\underline{e^3 - 1}}$$

Q6b S11

composition of functions

"inner derivative"

$4x^3$

Determine the following integral:

$$\int x^3 \cos(x^4 - 1)\, dx$$

Let us introduce the new variable t by

$t = x^4 - 1 \qquad \frac{dt}{dx} = 4x^3 \qquad dt = 4x^3\,dx,$

$\frac{1}{4}dt = x^3\,dx$

$$\int x^3 \cos(x^4-1)\,dx = \int \cos t \cdot \frac{1}{4}\,dt = \frac{1}{4}\int \cos t\,dt$$

$$= \frac{1}{4}\sin t + C = \underline{\frac{1}{4}\sin(x^4-1) + C}$$

Q6c S11

Determine the following integral:

$$\int \frac{x+4}{x^2-x-2}\,dx$$

① Make sure degree of numerator < degree of denominator
(true!)

② Factorise the denominator:
$x^2 - x - 2 = (x+1)(x-2)$

③ Solve $\frac{x+4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$ for A and B

Q6c S11

Determine the following integral:

$$\int \frac{x+4}{x^2-x-2}\,dx$$

$$\frac{x+4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$ multiply by $(x+1)(x-2)$

$$x+4 = A(x-2) + B(x+1)$$

$$x+4 = Ax - 2A + Bx + B$$

$$1 \cdot x + 4 = (A+B) \cdot x + (B-2A)$$

$$\begin{cases} A+B = 1 \\ B-2A = 4 \end{cases} \qquad 3B = 6 \Rightarrow B = 2 \Rightarrow A = -1$$

so $\begin{cases} A = -1 \\ B = 2 \end{cases}$

Q6c S11

Determine the following integral:

$$\int \frac{x+4}{x^2-x-2}\,dx$$

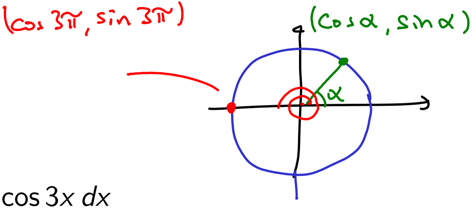
Thus $$\frac{x+4}{x^2-x-2} = -\frac{1}{x+1} + \frac{2}{x-2}$$

(4) Integrate!

$$\int \frac{x+4}{x^2-x-2}\,dx = \int -\frac{1}{x+1} + \frac{2}{x-2}\,dx =$$

$$= -\ln|x+1| + 2\ln|x-2| + C.$$

Q6d S11



Determine the following integral:

$$\int_0^\pi x^2 \cos 3x \, dx$$

Let us use integration by parts:

$$\int_0^\pi x^2 \cdot \cos 3x \, dx = \left[x^2 \cdot \frac{1}{3} \sin 3x \right]_0^\pi - \int_0^\pi 2x \cdot \frac{1}{3} \sin 3x \, dx$$

$$= \underbrace{\pi^2 \cdot \frac{1}{3} \sin 3\pi}_{=0} - \underbrace{0^2 \cdot \frac{1}{3} \sin 0}_{=0} - \frac{2}{3} \int_0^\pi x \cdot \sin 3x \, dx =$$

$$= -\frac{2}{3} \left[x \cdot \frac{-1}{3} \cos 3x \right]_0^\pi + \frac{2}{3} \int_0^\pi -\frac{1}{3} \cos 3x \, dx =$$

Q6d S11

Determine the following integral:

$$\int_0^\pi x^2 \cos 3x \, dx$$

$$= \frac{2}{9}\Big[x \cos 3x\Big]_0^\pi - \frac{2}{9}\int_0^\pi \cos 3x \, dx =$$

$$= \frac{2}{9}\Big(\pi \cdot \cos 3\pi - \underbrace{0 \cdot \cos 0}_{=0}\Big) - \frac{2}{9}\Big[\frac{1}{3}\sin 3x\Big]_0^\pi =$$

$$= \frac{2}{9} \cdot \pi \cdot (-1) - \frac{2}{9}\Big(\underbrace{\frac{1}{3}\sin 3\pi}_{=0} - \underbrace{\frac{1}{3}\sin 0}_{=0}\Big)$$

$$= -\frac{2\pi}{9}$$

Q7a S11

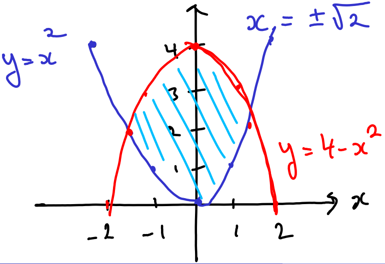
Calculate the area of the region of the plane that is enclosed by the parabolas $y = x^2$ and $y = 4 - x^2$.

We begin by finding the points of intersection between the parabolas: by solving

$$x^2 = 4 - x^2$$
$$2x^2 = 4$$
$$x^2 = 2$$
$$x = \pm\sqrt{2}$$

A table of function values

x	x^2	$4-x^2$
-2	4	0
$-\sqrt{2}$	2	2
-1	1	3
0	0	4
1	1	3
$\sqrt{2}$	2	2
2	4	0



Q7a S11

$$(\sqrt{2})^3 = \sqrt{2}\cdot\sqrt{2}\cdot\sqrt{2} = 2\sqrt{2}$$

Calculate the area of the region of the plane that is enclosed by the parabolas $y = x^2$ and $y = 4 - x^2$.

The sought area is equal to

$$\int_{-\sqrt{2}}^{\sqrt{2}} (4-x^2) - x^2 \, dx = \int_{-\sqrt{2}}^{\sqrt{2}} 4 - 2x^2 \, dx =$$

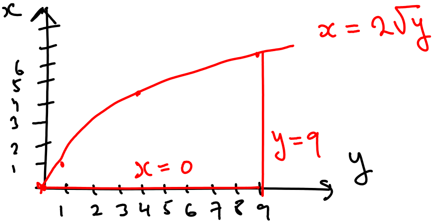
$$= \left[4x - \frac{2}{3}x^3\right]_{-\sqrt{2}}^{\sqrt{2}} = 4\sqrt{2} - \frac{2}{3}(\sqrt{2})^3 - \left(4(-\sqrt{2}) - \frac{2}{3}(-\sqrt{2})^3\right)$$

$$= 4\sqrt{2} - \frac{4}{3}\sqrt{2} + 4\sqrt{2} - \frac{4}{3}\sqrt{2} = \left(8 - \frac{8}{3}\right)\sqrt{2} = \frac{16}{3}\sqrt{2}$$

Q7b S11

Calculate the volume of the solid obtaided by rotating the region that is bounded by the curves $x = 2\sqrt{y}$, $x = 0$ and $y = 9$ about the y-axis.

Let us first draw a sketch:



y	$2\sqrt{y}$
0	0
1	2
4	4
9	6

The volume of the solid is given by the integral

$$\int_0^9 \pi \cdot r(y)^2 \, dy = \int_0^9 \pi \left(2\sqrt{y}\right)^2 dy = \pi \int_0^9 4y \, dy = \pi \left[2y^2\right]_0^9 = \pi \cdot 2 \cdot 9^2 - 0 = 162\pi$$

Q8a S11

Find the solution of the differential equation $2xy' + y = 6x$, $x > 0$ and $y(2) = 20$.

Not covered this year!

Q8b S11

The number of P of dung beetles at the time t (measured in weeks) in a population with limited resources satisfies the *logistic differential equation*

$$\frac{dP}{dt} = 0.2P - 0.00004P^2.$$

(i) What is the carrying capacity?

(ii) If the population at the beginning of May this year was 200 dung beetles, what will the population be after eight weeks?

I.e. Solve $\begin{cases} \dfrac{dP}{dt} = 0.2P - 0.00004P^2 \\ P(0) = 200 \end{cases}$ Find $P(8)$

The logistic differential equation is usually written as $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$ where K is the carrying capacity

$$\frac{dP}{dt} = 0.2P - 0.00004P^2$$
$$= 0.2P\left(1 - \frac{P}{K}\right)$$

$$\frac{dP}{dt} = k \cdot P \cdot \left(1 - \frac{P}{K}\right)$$

so $\quad 0.2P - 0.00004\,P^2 = 0.2P - 0.2\,\frac{P^2}{K}$

so $\quad \frac{0.2}{K} = 0.00004 \implies K = \frac{0.2}{0.00004} = 50000$

(a) <u>Answer</u>: The carrying capacity is 50000

We know that the solution to the logistic eq is

$$P(t) = \frac{K}{1 + A e^{-kt}}$$

In our case $\quad K = 50000$

$k = 0.2$

$$A = \frac{K - P_0}{P_0} = 249.$$

Thus, the solution to our initial value problem is

$$P(t) = \frac{50000}{1+249e^{-0.2t}}$$

So the population after eight weeks is

$$P(8) = \frac{50000}{1+249e^{-1.6}} = 975.19,$$ so

975 dung beetles