The definite integral

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MA100

The definition of the definite integral

Inspired by the definition of area below the graph of a function from the last lecture, we make the following definition.

Definition

If f is a function defined on $[a, b]$, we define *the definite integral of f from a to b* to be

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta_x$$

where

- $\Delta_x = (b-a)/n$
- $x_i = a + i\Delta_x$, so $a = x_0$ and $b = x_n$
- $x_i^* \in [x_{i-1}, x_i]$.

provided that the limit exists. If the limit exists, the function f is said to be *integrable* on $[a, b]$.

The definition of the definite integral

In the expression

$$\int_a^b f(x)\,dx$$

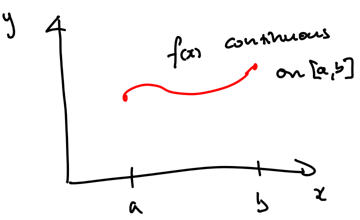
- $\int$ is the *integral sign*
- a and b are the *limits of integration* (a is the lower, and b is the upper).
- $f(x)$ is the *integrand*

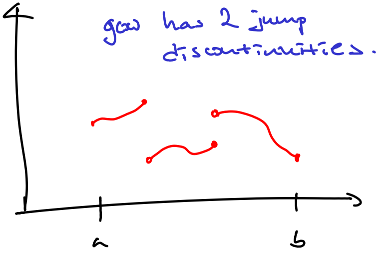
Integrable functions

Theorem

If f is continuous[a] *on $[a, b]$ then f is integrable on $[a, b]$, i.e. the definite integral $\int_a^b f(x)\,dx$ exists.*

[a] or if f has only a finite number of jump discontinuities





The integral

If f is continuous[1] on the interval $[a, b]$, then the actual points in the intervals $[x_{i-1}, x_i]$ we use does not matter, and we can for simplicity choose the right endpoint, giving us the following result.

Theorem

If $f(x)$ is continuous[a] on the interval $[a, b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta_x$$

where

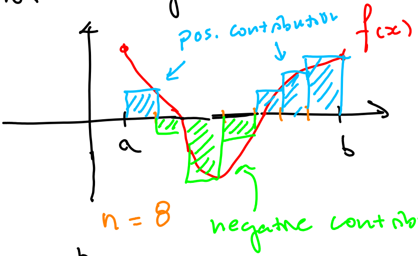
$$\Delta_x = \frac{b-a}{n} \qquad \text{and } x_i = a + i\Delta_x.$$

[a] or integrable

[1] or, more generally, integrable

Net area

Suppose f is continuous on $[a,b]$, but not always positive

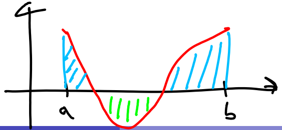


Now:

$$\int_a^b f(x)\,dx = \lim \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\,\Delta x$$

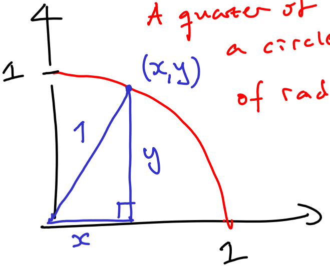
So $\int_a^b f(x)\,dx$ will be the net area or signed area bounded by the graph



$$\int_a^b f(x)\,dx = \text{Blue area} - \text{Green Area}$$

Interpreting integrals in terms of areas

Determine the definite integral $\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$



A quarter of a circle of radius 1

By Pythagoras:

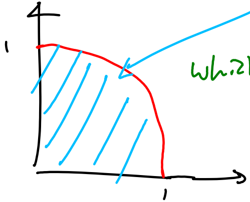
$$1^2 = x^2 + y^2$$

$$y^2 = 1 - x^2$$

$$y = \sqrt{1-x^2}$$

So:

$\displaystyle\int_0^1 \sqrt{1-x^2}\,dx$ is then the area of shaded region



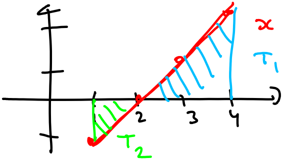
which is $\dfrac{\pi}{4}$

Interpreting integrals in terms of areas (cont.)

Evaluate the definite integral

$$\int_1^4 x-2 \, dx$$

Draw the graph



Now

$$\int_1^4 x-2 \, dx =$$

Area of T_1 − Area of T_2

$$= \frac{2\cdot 2}{2} - \frac{1\cdot 1}{2} = \frac{3}{2}$$

Theorem

Let $f(x)$ be a continuous[a] function. Then

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

[a] integrable

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x =$$

$$-\lim_{n\to\infty} - \sum_{i=1}^{n} f(x_i)\Delta x$$

$$\Delta x = \frac{b-a}{n} \qquad x_i = a + i\Delta x$$

$$-\int_b^a f(x)\,dx = -\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i')\Delta x'$$

$$\Delta x' = \frac{a-b}{n} = -\Delta x$$

Properties of the integral

Theorem

Let f and g be continuous functions and c a constant. Then

$$\int_a^b c\,dx = c(b-a) \tag{1}$$

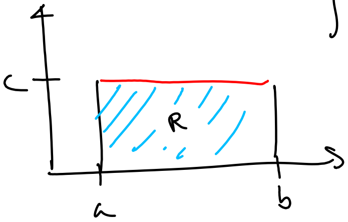
$$\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx \tag{2}$$

$$\int_a^b c\cdot f(x)\,dx = c\int_a^b f(x)\,dx \tag{3}$$

$$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \tag{4}$$

Properties of the integral (cont.)

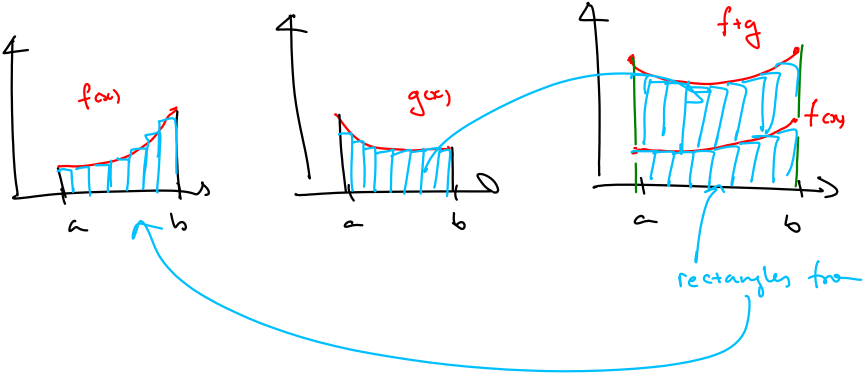
$$\int_a^b c\,dx = c(b-a), \quad c \text{ is a constant.}$$



$\int c\,dx = \text{Area of } R = c(b-a)$

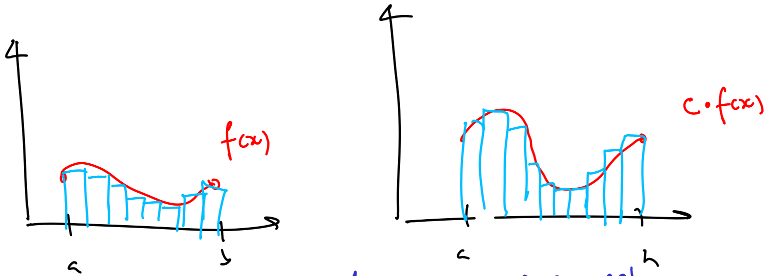
Properties of the integral (cont.)

$$\int_a^b (f(x) + g(x))\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$



Properties of the integral (cont.)

$$\int_a^b c \cdot f(x)\, dx = c \int_a^b f(x)\, dx$$



The area of each rectangle on the right is c times the area of corresponding rectangle on the left

Properties of the integral (cont.)

$$\int_a^b (f(x) - g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

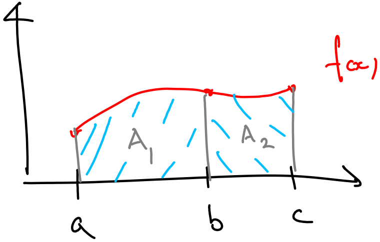
$$\int_a^b (f(x) - g(x))\,dx = \int_a^b \big(f(x) + (-1)\cdot g(x)\big)\,dx$$

$$= \int_a^b f(x) + \int_a^b (-1)\cdot g(x)\,dx = \int_a^b f(x)\,dx + (-1)\int_a^b g(x)\,dx$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

Properties of the integral (cont.)

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$



$$\int_a^c f(x)\,dx = \text{Area of } A_1 + \text{Area of } A_2 = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$