The fundamental theorem of calculus

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Comparison properties of the integral

Theorem

Let $f(x)$ and $g(x)$ be continuous[a]*, then*

- *If $f(x) \geq g(x)$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

- *If $f(x) \geq 0$ for all $x \in [a, b]$, then*

$$\int_a^b f(x)\,dx \geq 0.$$

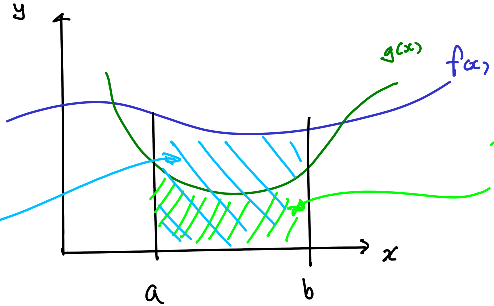
- *If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then*

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$

[a] or integrable

If $f(x) \geq g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

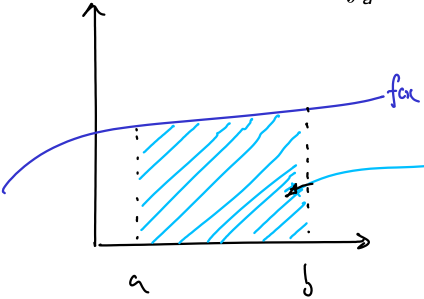


The blue area is $\int_a^b f(x)\,dx$

This area is $\int_a^b g(x)\,dx$

If $f(x) \geq 0$ for all $x \in [a, b]$, then

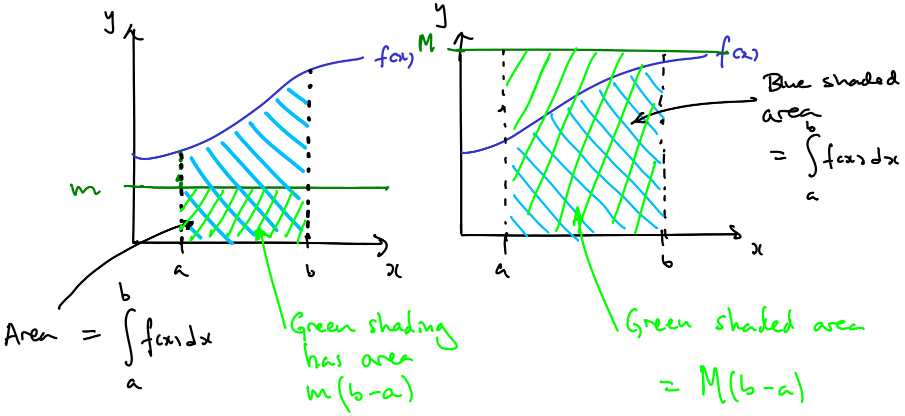
$$\int_a^b f(x)\,dx \geq 0.$$



$\int_a^b f(x)\,dx =$ Area of region

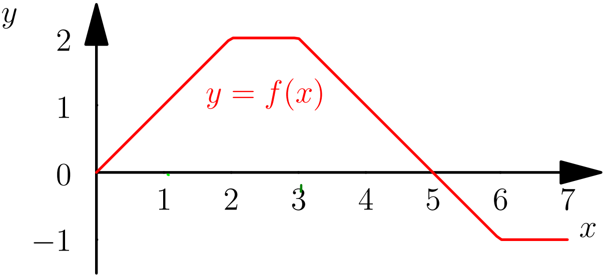
If $m \leq f(x) \leq M$ for all $x \in [a, b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a).$$



Example

Sketch a graph of $g(x) = \int_0^x f(t)\,dt$ where



Let us calculate some values of $g(x)$

$g(0) = \int_0^0 f(t)\,dt = 0$. $\quad g(1) = \int_0^1 f(x)\,dx = \frac{1}{2}$ $\quad g(7) = \frac{11}{2} - 1 = \frac{9}{2}$

$g(2) = \int_0^2 f(t)\,dt = \frac{2\cdot 2}{2} = 2$ $\quad g(3) = \int_0^3 f(t)\,dt = 2+2 = 4$

$g(5) = 2+2+2 = 6$ $\quad g(4) = 6 - \frac{1}{2} = \frac{11}{2}$ $\quad g(6) = 6 - \frac{1}{2} = \frac{11}{2}$

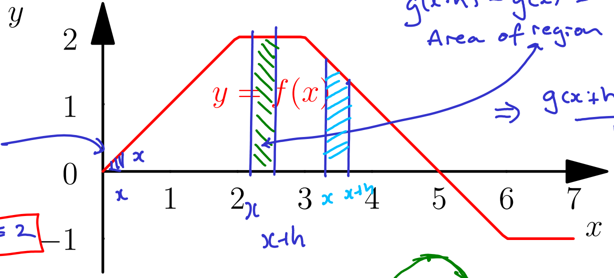
Example

Sketch a graph of $g(x) = \int_0^x f(x)\,dx$ where

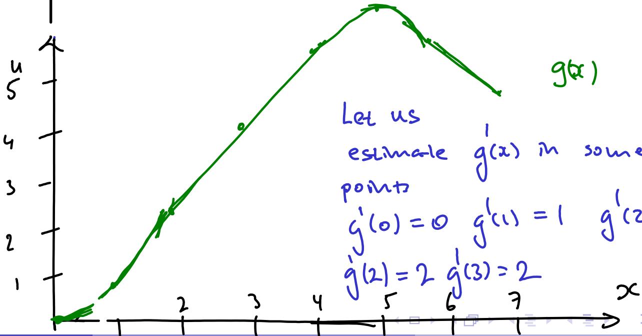
$g(x+h) - g(x) =$ Area of region $= 2h$ $\boxed{2 \le x \le 3}$

$\Rightarrow \dfrac{g(x+h) - g(x)}{h} = 2$

Area $\dfrac{x^2}{2} = g(x)$ for $\boxed{0 \le x \le 2}$



x	$g(x)$
0	0
1	$1/2$
2	2
3	4
4	$11/2$
5	6
6	$11/2$
7	$9/2$



Let us estimate $g'(x)$ in some points

$g'(0) = 0 \quad g'(1) = 1 \quad g'(2) = 2$

$g'(2) = 2 \quad g'(3) = 2$

The fundamental theorem of calculus part 1

Theorem

If f is continuous on $[a, b]$, then the function g defined by

$$g(x) = \int_a^x f(t)\, dt$$

is continuous on $[a, b]$, and differentiable on (a, b) with

$$g'(x) = f(x).$$

Example

Find the derivative of the function $g(x) = \int_0^x \sin(\sqrt{t})\, dt$.

The function $f(t) = \sin\sqrt{t}$ is continuous, so, therefore, by the fundamental theorem of calculus

$$g'(x) = \sin\sqrt{x}$$

The fundamental theorem of calculus part 2

Theorem

If f is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

The fundamental theorem of calculus part 2 (cont)

Let $g(x) = \int_a^x f(t)\,dt$ Then, by part 1

of the fundamental theorem of calculus,

$$g'(x) = f(x)$$

If $F(x)$ is any other antiderivative of f,

$$F(x) = g(x) + C \quad , \; C \text{ constant.}$$

Now :

$$\boxed{g(a) = \int_a^a f(t)\,dt = 0}$$

$$\begin{aligned} F(b) - F(a) &= g(b) + C - (g(a) + C) \\ &= g(b) - g(a) \\ &= g(b) \\ &= \int_a^b f(t)\,dt = \int_a^b f(x)\,dx \end{aligned}$$

What if f is not continuous?

Can we evaluate

$$\int_{-2}^{2} \frac{1}{x^4}\, dx?$$

If we forgot to check the continuity of $\frac{1}{x^4}$ on $[-2,2]$ we might get

"$$\int_{-2}^{2} \frac{1}{x^4}\, dx = F(2) - F(-2)$$

(where $F(x) = -\frac{1}{3}x^{-3} = -\frac{1}{3x^3}$)

$$= -\frac{1}{3 \cdot 2^3} - \left(-\frac{1}{3(-2)^3}\right) = -\frac{1}{24} - \frac{1}{24} = -\frac{1}{12}$$"

$\frac{1}{x^4} = x^{-4}$

so $\frac{d}{dx} \frac{x^{-3}}{-3} = \frac{(-3)}{(-3)} \cdot x^{-3-1} = x^{-4}$

False! See next slide.

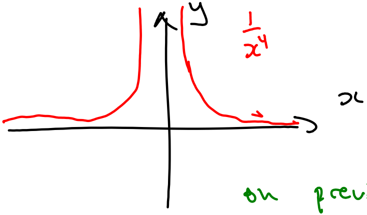
What if f is not continuous?

Can we evaluate

$$\int_{-2}^{2} \frac{1}{x^4}\, dx?$$

Since $\frac{1}{x^4} \geq 0$, $\underbrace{\int_{-2}^{2} \frac{1}{x^4}\, dx}_{\text{if it exists}} \geq 0$.

This contradicts our calculation on previous slide



From the graph we see that the integrand $\frac{1}{x^4}$ is not continuous. Hence, calculations on previous slide nonsense!

What if f is not continuous?

Can we evaluate

$$\int_{-2}^{2} \frac{1}{x^4}\,dx?$$

$\frac{1}{x^4}$ is continuous on the interval $x > 0$ though. So, if we instead wanted to calculate $\int_1^2 \frac{1}{x^4}\,dx$ we would get

$$\int_1^2 \frac{1}{x^4}\,dx = \underbrace{\left[-\frac{1}{3x^3}\right]_1^2}_{} = -\frac{1}{3\cdot 2^3} - \left(-\frac{1}{3\cdot 1^3}\right)$$

$$= \frac{-1}{3\cdot 8} + \frac{1}{3} = \frac{7}{24}$$

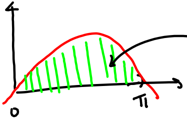
Notation $[F(x)]_a^b = F(b) - F(a)$

Example

Evaluate the integral $\int_0^{\pi} \sin(x)\,dx$.

The function $\sin x$ is continuous on $\mathbb{R}$

Thus, we can use the fundamental theorem of calculus to get:



$$\int_0^{\pi} \sin x \, dx = $$

$$= \Big[-\cos x\Big]_0^{\pi} =$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\frac{d}{dx} -\cos x = --\sin x = \sin x$$

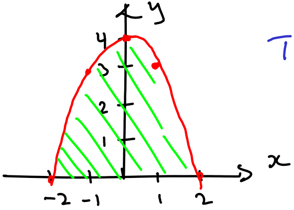
so $-\cos x$ is an antiderivative of $\sin x$

$$= -\cos \pi - (-\cos 0) = -(-1) - (-1) = \underline{2}$$

Example

Find the area below the parabola $y = 4 - x^2$ between $x = -2$ and $x = 2$.

x	y
-2	0
-1	3
0	4
1	3
2	0



The area of the shaded region is

$$\int_{-2}^{2} (4 - x^2)\, dx$$

$4 - x^2$ is continuous, so by the fund. thm. of calc.

$$\int_{-2}^{2} 4 - x^2 \, dx = \left[4x - \frac{x^3}{3} \right]_{-2}^{2} = \left(4 \cdot 2 - \frac{2^3}{3}\right) - \left(4(-2) - \frac{(-2)^3}{3}\right)$$

$$= \frac{16}{3} - \left(-\frac{16}{3}\right) = \frac{32}{3}$$

Example

Find $g'(x)$ when

$$g(x) = \int_1^{x^3} \sin\sqrt{t}\, dt$$

Now we have a composition of functions.

If $F(x)$ is an antiderivative of $\sin\sqrt{x}$

then $\int_1^x \sin\sqrt{t}\, dt = F(x) - F(1)$

$F'(x) = \sin\sqrt{x}$

So $g(x) = \int_1^{x^3} \sin\sqrt{t}\, dt = F(x^3) - F(1)$

Then, by the chain rule $g'(x) = F'(x^3)\cdot 3x^2 = \sin\sqrt{x^3}\cdot 3x^2$

$= 3x^2 \cdot \sin x^{3/2}$

The fundamental theorem of calculus (both parts)

Theorem

Let $f(x)$ be continuous on $[a, b]$, then:

1. *If* $g(x) = \int_a^x f(t)\,dt$, *then* $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$, *for* $F(x)$ *any antiderivative of* $f(x)$.