Indefinite integrals and the net change theorem

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Indefinite integrals

We will use the following notation

$$\int f(x)\,dx = F(x) \qquad \text{means} \qquad F'(x) = f(x)$$

So, for example

$$\int x\,dx = \frac{x^2}{2} + \underline{C}$$

and

$$\int e^x\,dx = e^x + \underline{C}$$

constant of integration

Example

Verify that

$$\int \frac{x}{\sqrt{x^2+1}}\,dx = \sqrt{x^2+1} + C$$

We need to check that

$$\frac{d}{dx}\left(\sqrt{x^2+1} + C\right) = \frac{x}{\sqrt{x^2+1}}$$

, so we differentiate:

$$\frac{d}{dx}\left(\sqrt{x^2+1} + C\right) = \frac{d}{dx}\sqrt{x^2+1} = \frac{d}{dx}\left(x^2+1\right)^{1/2} =$$

$$= \frac{1}{2}\left(x^2+1\right)^{\frac{1}{2}-1}\cdot 2x = x\left(x^2+1\right)^{-\frac{1}{2}} = \frac{x}{(x^2+1)^{1/2}} = \frac{x}{\sqrt{x^2+1}}$$

Table of indefinite integrals

$$\int c\, f(x)\, dx = c \int f(x)\, dx$$

$$\int k\, dx = kx + C$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \frac{1}{x}\, dx = \ln|x| + C$$

$$\int e^x\, dx = e^x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \frac{1}{1+x^2}\, dx = \tan^{-1} x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C$$

Example

N.B $\cos^3 x = (\cos x)^3$

Verify that

$$\int \cos^3 x\, dx = \sin x - \frac{1}{3}\sin^3 x + C$$

Again, we need to check that

$$\frac{d}{dx}\left(\sin x - \frac{1}{3}\sin^3 x + C\right) = \cos^3 x$$

We differentiate to get:

$$\frac{d}{dx}\left(\sin x - \frac{1}{3}\sin^3 x + C\right) = \cos x - \frac{1}{3}\cdot 3\sin^2 x \cdot \cos x$$

$$= \cos x - \cos x \cdot \sin^2 x = \cos x \underbrace{\left(1 - \sin^2 x\right)}_{\cos^2 x}$$

$$= \cos x \cdot \cos^2 x = \cos^3 x$$

Example

Find the indefinite integral $\int (x^5 + 12x^2)\,dx$

$$\int x^5 + 12x^2\,dx = \int x^5\,dx + \int 12x^2\,dx =$$

$$= \frac{x^6}{6} + C_1 + 12\int x^2\,dx = \frac{x^6}{6} + C_1 + 12\left(\frac{x^3}{3} + C_2\right)$$

$$= \frac{x^6}{6} + 4x^3 + \underbrace{C_1 + 12C_2}_{C} = \frac{x^6}{6} + 4x^3 + C$$

Example

Find the indefinite integral $\int (x^5 + 12x^2)\, dx$

Example

Evaluate $\int_1^4 \sqrt{t}(1+t)\,dt$

Here we have a definite integral.
i.e. the answer will be
a number.

$$\int_1^4 \sqrt{t}\,(1+t)\,dt = \int_1^4 \left(\sqrt{t} + t\sqrt{t}\right)dt = \int_1^4 \left(t^{1/2} + t^{3/2}\right)dt$$

$$= \left[\frac{t^{1/2+1}}{1/2+1} + \frac{t^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right]_1^4 = \left[\frac{t^{3/2}}{3/2} + \frac{t^{5/2}}{5/2}\right]_1^4 = \left[\frac{2}{3}t^{3/2} + \frac{2}{5}t^{5/2}\right]_1^4$$

$$\left[\frac{2}{3}t\sqrt{t} + \frac{2}{5}t^2\sqrt{t}\right]_1^4 = \left(\frac{2}{3}4\sqrt{4} + \frac{2}{5}4^2\sqrt{4}\right) - \left(\frac{2}{3}1\sqrt{1} + \frac{2}{5}1^2\sqrt{1}\right)$$

$$= \frac{16}{3} + \frac{64}{5} - \frac{2}{3} - \frac{2}{5} = \frac{14}{3} + \frac{62}{5} = \frac{276}{15}$$

Example

Evaluate $\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\, dt$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

$$\tan 60^\circ = \frac{\sin 60^\circ}{\cos 60^\circ} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}}$$

$$\int_0^{1/\sqrt{3}} \frac{t^2-1}{t^4-1}\, dt = \int_0^{1/\sqrt{3}} \frac{t^2-1}{(t^2+1)(t^2-1)}\, dt = \int_0^{1/\sqrt{3}} \frac{1}{t^2+1}\, dt$$

$$= \left[\tan^{-1} t\right]_0^{1/\sqrt{3}} = \tan^{-1}\frac{1}{\sqrt{3}} - \tan^{-1} 0 = \frac{\pi}{3} - 0 = \frac{\pi}{3}$$

The net change theorem

Theorem

The integral of a rate of change is the net change.

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

Example

pos. velocity

neg velocity

A particle moves along a line so that its velocity at time t is $v(t) = t^2 - t - 6$ measured in metres per second.

1. Find the displacement of the particle during the time period $1 \leq t \leq 4$.
2. Find the distance travelled during this time period.

(1) By the net change theorem we get the net change of position by integrating the velocity:

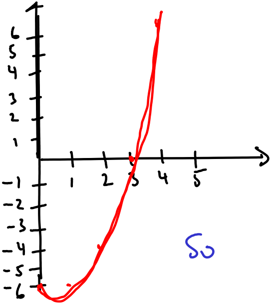
$$s(4) - s(1) = \int_1^4 s'(t)\,dt = \int_1^4 v(t)\,dt = \int_1^4 (t^2 - t - 6)\,dt$$

$$= \left[\frac{t^3}{3} - \frac{t^2}{2} - 6t\right]_1^4 = \left(\frac{4^3}{3} - \frac{4^2}{2} - 6\cdot 4\right) - \left(\frac{1}{3} - \frac{1}{2} - 6\right) =$$

$$= \frac{64}{3} - 8 - 24 - \frac{1}{3} + \frac{1}{2} + 6 = -\frac{16}{3}$$

Example (cont.)

(2) Distance travelled : $\int_1^4 |v(t)|\,dt$



$v(t) = t^2 - t - 6$

$1 \le t \le 4$

t	$v(t)$
0	-6
1	-6
2	-4
3	0
4	6

So $v(t) \ge 0$ for $t \ge 3$

$v(t) \le 0$ for $1 \le t \le 3$

$$\Rightarrow |v(t)| = \begin{cases} t^2 - t - 6 & 3 \le t \le 4 \\ -t^2 + t + 6 & 1 \le t \le 3 \end{cases} \quad \text{so}$$

$$\int_1^4 |t^2 - t - 6|\,dt = \int_1^3 |t^2 - t - 6|\,dt + \int_3^4 |t^2 - t - 6|\,dt =$$

Example (cont.)

$$= \int_1^3 -t^2+t+6\,dt + \int_3^4 t^2-t-6\,dt =$$

$$= \left[-\frac{t^3}{3}+\frac{t^2}{2}+6t\right]_1^3 + \left[\frac{t^3}{3}-\frac{t^2}{2}-6t\right]_3^4 =$$

$$= \left(-\frac{27}{3}+\frac{9}{2}+18\right) - \left(-\frac{1}{3}+\frac{1}{2}+6\right) + \left(\frac{64}{3}-\frac{16}{2}-24\right)$$

$$-\left(\frac{27}{3}-\frac{9}{2}-18\right) = \frac{11}{3}+\frac{1}{2}+6 = 6+\frac{25}{6} =$$

$$= 10\frac{1}{6}$$