The substitution rule

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Motivating example

Find

$$\int \underline{\underline{2x}}\,\underline{\underline{\sqrt{x^2+1}}}\,dx$$

Set u (a new variable) to be x^2+1

then $\frac{du}{dx} = 2x$, so we might write

"$2x\,dx = du$" so

$$\int \underbrace{\sqrt{x^2+1}}_{\sqrt{u}}\;\underbrace{2x\,dx}_{du} = \int \sqrt{u}\,du = \int u^{1/2}\,du = \frac{u^{3/2}}{3/2} + C = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^2+1)^{3/2} + C$$

The substitution rule

Theorem

Let I be an interval where $u = g(x)$ is differentiable and $f(x)$ continuous. Then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If $F' = f$,

$$\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C$$

since $\frac{d}{dx}\left[F(g(x)) + C\right] = F'(g(x)) \cdot g'(x)$

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C = F(u) + C = \int F'(u)\,du$$

Setting $f = F'$ gives the statement

Example

Find

$$\int e^{-3x}\,dx$$

Let $u = -3x$, so $\frac{du}{dx} = -3$; $du = -3dx$

$$\int e^{\overbrace{-3x}^{u}}\,dx = \int -\frac{1}{3}\underbrace{e^{-3x}}_{e^u}\cdot\underbrace{(-3)dx}_{du} = -\frac{1}{3}\int e^{u}\,du$$

$$= -\frac{1}{3}e^{u} + C = -\frac{1}{3}e^{-3x} + C$$

Example

Find
$$\int x^3 \cos(x^4 + 2)\, dx$$

Let $u = x^4 + 2$, so $\frac{du}{dx} = 4x^3$, $du = 4x^3\,dx$

Using this substitution, we get:

$$\int x^3 \cos(x^4+2)\,dx = \int \frac{1}{4} \underbrace{\cos(x^4+2)}_{\cos u} \cdot \underbrace{4x^3\,dx}_{du} =$$

$$= \frac{1}{4}\int \cos u\,du = \frac{1}{4}\cdot \sin u + C = \frac{1}{4}\sin(x^4+2) + C$$

Example

Find

$$\int \tan x \, dx$$

We have that $\tan x = \dfrac{\sin x}{\cos x}$. Let $u = \cos x$

So $\dfrac{du}{dx} = -\sin x$, so $du = -\sin x \, dx$, so:

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int \frac{-1}{\cos x} \cdot (-\sin x) \, dx$$

$$= -\int \frac{1}{u} \, du = -\ln|u| + C = -\ln|\cos x| + C$$

The substitution rule for definite integrals

Theorem

If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Let $F' = f$

$$\int_a^b f(g(x))g'(x)\,dx = \Big[F(g(x))\Big]_a^b = F(g(b)) - F(g(a))$$

$$\shortparallel$$

$$\int_{g(a)}^{g(b)} f(u)\,du = \Big[F(u)\Big]_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$$

Example

Evaluate

$$\int_0^4 \sqrt{2x+1}\,dx$$

Let $u = 2x+1$ so $\frac{du}{dx} = 2$, so $du = 2dx$

Therefore, we have:

$x = 0 \Rightarrow u = 2\cdot 0 + 1 = 1$

$x = 4 \Rightarrow u = 2\cdot 4 + 1 = 9$

$$\int_0^4 \sqrt{2x+1}\,dx = \int_0^4 \frac{1}{2}\underbrace{\sqrt{2x+1}}_{\sqrt{u}}\cdot\underbrace{2dx}_{du} = \frac{1}{2}\int_{\underline{\underline{1}}}^{\underline{\underline{9}}} \sqrt{u}\,du$$

$$= \frac{1}{2}\int_1^9 u^{1/2}\,du = \frac{1}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^9 = \frac{1}{3}\left[u^{3/2}\right]_1^9 = \frac{1}{3}\left(9^{3/2} - 1^{3/2}\right) = \frac{26}{3}$$