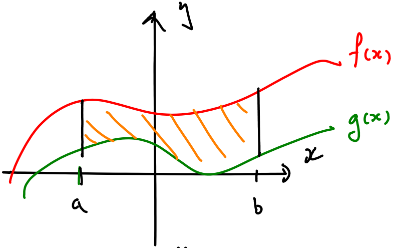
Areas between curves

E. Sköldberg
emil.skoldberg@nuigalway.ie
http://www.maths.nuigalway.ie/~emil/

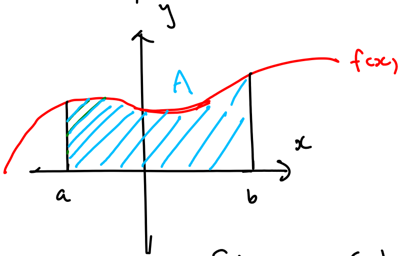
School of Mathematics etc.
National University of Ireland, Galway

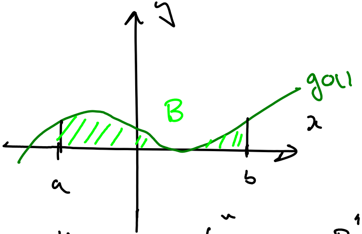
MA100

The area between two curves



Problem: How to calculate the area of the shaded region?





Solution: subtract the area of "region B" from the area of "region A"

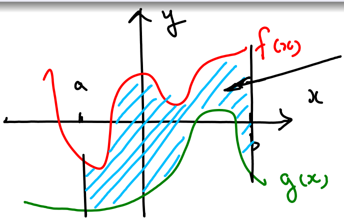
i.e $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx = \int_a^b (f(x) - g(x))\,dx$

A formula for the area

Theorem

The area A of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$ and $y = b$ where f and g are continuous and $f(x) \geq g(x)$ for all $x \in [a, b]$ is

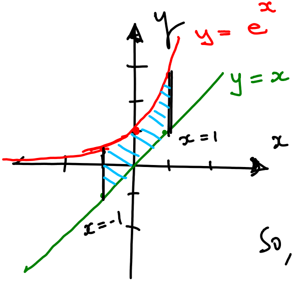
$$A = \int_a^b [f(x) - g(x)]\, dx$$



Area $= \int_a^b (f(x) - g(x))\, dx$

Example

Find the area of the region bounded above by $y = e^x$, below by $y = x$, from the left by $x = -1$ and on the right by $x = 1$.



We know that (and can see in the graph!) that

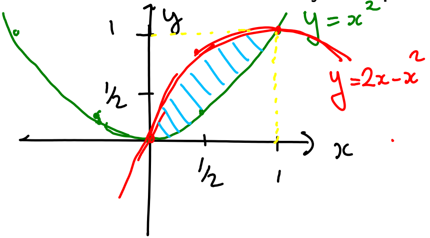
$x \le e^x$ for all x.

So, by the theorem on the previous slide, the area is $\displaystyle\int_{-1}^{1} (e^x - x)\,dx =$

$$= \left[e^x - \frac{x^2}{2}\right]_{-1}^{1} = \left(e^1 - \frac{1}{2}\right) - \left(e^{-1} - \frac{1}{2}\right) = e^1 - e^{-1} = e - \frac{1}{e}$$

Example

Find the area enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$.



x	x^2	$2x - x^2$
-2	4	-8
-1	1	-3
0	0	0
1	1	1
2	4	0
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{3}{4}$

(1) We find the points of intersection between the curves:

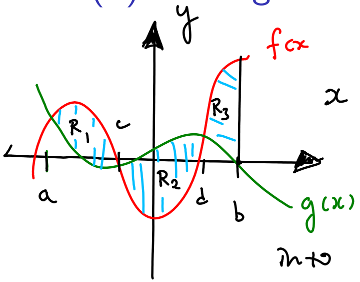
$$x^2 = 2x - x^2$$
$$2x^2 = 2x$$
$$x^2 = x \Rightarrow \begin{cases} x = 0 \\ x = 1 \end{cases}$$

(2) Between $x = 0$ & $x = 1$

$2x - x^2 \geqslant x^2$, so the area is:

$$\int_0^1 (2x - x^2 - x^2)\,dx = \int_0^1 2x - 2x^2\,dx = \left[x^2 - \frac{2x^3}{3}\right]_0^1 = 1 - \frac{2}{3} - 0 = \frac{1}{3}$$

When $f(x)$ is not greater than $g(x)$.



If $f(x)$ is not always greater than $g(x)$, we divide the region into smaller regions, and calculate their area separately: and then sum!

Area of R_1: For $a \leq x \leq c$, $f(x) \geq g(x)$ so.

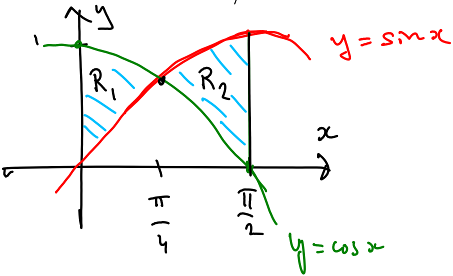
Area of $R_1 = \int_a^c f(x) - g(x)\, dx$

Area of R_2: For $c \leq x \leq d$ $g(x) \geq f(x)$ so the area is $\int_c^d g(x) - f(x)\, dx$

Area of R_3: In the same way as R_1: $\int_d^b f(x) - g(x)\, dx$

Example

Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \pi/2$.



① Graph

② Find point(s) of intersection:

Solve $\sin x = \cos x$

$\frac{\sin x}{\cos x} = 1$

$\tan x = 1$

$\Rightarrow$

$x = \frac{\pi}{4}$

③ Integrate:

Area of R_1 =

$$\int_0^{\pi/4} \cos x - \sin x \, dx = \Big[\sin x + \cos x\Big]_0^{\pi/4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - (0+1) = \sqrt{2} - 1$$

Integration with respect to y

Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.

Area of R_2 : $\displaystyle\int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx =$

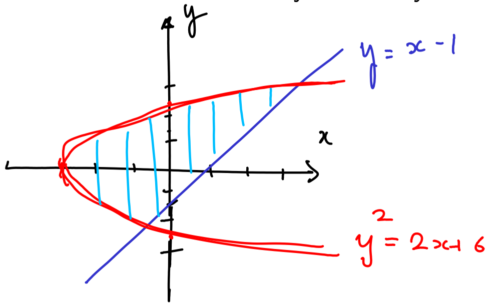
$$= \Big[- \cos x - \sin x \Big]_{\pi/4}^{\pi/2} = (0 - 1) - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \right) = \sqrt{2} - 1$$

So, the total area is Area(R_1) + Area(R_2) =

$$= \sqrt{2} - 1 + \sqrt{2} - 1 = \underline{2\sqrt{2} - 2}$$

Integration with respect to y

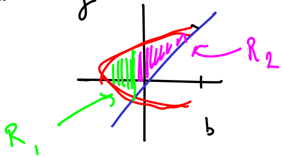
Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$.



$y^2 = 2x + 6$

$y = \pm\sqrt{2x+6}$

Hard way



Area = Area(R_1) + Area(R_2) =

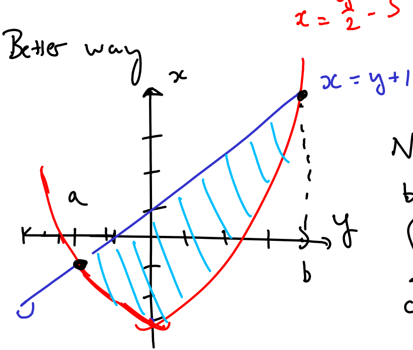
$$\int_{-3}^{?} \sqrt{2x+6} + \sqrt{2x+6}\, dx + \int_{?}^{b} \left(\sqrt{2x+6} - (x-1)\right) dx$$

Example (cont.)

$y^2 = 2x+6 \Rightarrow x = \frac{y^2}{2} - 3$

$y = x-1 \Rightarrow x = y+1$

Better way



Now the area is

$$\int_a^b (y+1) - \left(\frac{y^2}{2} - 3\right) dy$$

Find points of intersection by solving $\frac{y^2}{2} - 3 = y+1$

$y^2 - 2y - 8 = 0$

$(y-1)^2 - 9 = 0 \qquad (y-1)^2 = 9 \qquad \Rightarrow \begin{array}{l} a = -2 \\ b = 4 \end{array}$

$y - 1 = \pm 3 \qquad y = 4 \text{ or } -2$

Example (cont.)

So the area is

$$\int_{-2}^{4} (y+1) - \left(\frac{y^2}{2} - 3\right) dy = \int_{-2}^{4} -\frac{y^2}{2} + y + 4 \, dy$$

$$= \left[-\frac{y^3}{6} + \frac{y^2}{2} + 4y \right]_{-2}^{4} = \ldots$$