Noise and Multistability in the Square Root Map

Eoghan J. Staunton, Petri T. Piirainen

15 June 2018
The Square Root Map

Many impacting systems are described by a 1-D map known as the square root map near *grazing* impacts.

\[ x_{n+1} = S(x_n) = \begin{cases} \mu + bx_n & \text{if } x_n < 0, \\ \mu - a\sqrt{x_n} & \text{if } x_n \geq 0, \end{cases} \]

where \( a > 0 \) and \( b > 0 \).

Symbolically, if \( x_n < 0 \) it is represented by an \( L \) and if \( x_n > 0 \) it is represented by an \( R \).
The Period Adding Cascade

Here we will assume that the parameter $b$ (the slope of the linear part) is such that $0 < b < 1/4$. For values of $b$ in this range the deterministic square root map undergoes a period-adding cascade with intervals of multistability as the bifurcation parameter $\mu$ is decreased.
The Period Adding Cascade

Here we will assume that the parameter $b$ (the slope of the linear part) is such that $0 < b < 1/4$. For values of $b$ in this range the deterministic square root map undergoes a period-adding cascade with intervals of multistability as the bifurcation parameter $\mu$ is decreased.
The Period Adding Cascade

Here we will assume that the parameter $b$ (the slope of the linear part) is such that $0 < b < 1/4$. For values of $b$ in this range the deterministic square root map undergoes a period-adding cascade with intervals of multistability as the bifurcation parameter $\mu$ is decreased.

These periodic orbits take the form $(RL^m)^\infty$ for $m = 1, 2, 3, \ldots$. This means they consist of one iterate on the right ($> 0$) followed by $m$ iterates on the left ($< 0$).
Riddled Basins of Attraction

On regions of multistability the basins of attraction of the two periodic attractors have a complex *riddled* structure.
On regions of multistability the basins of attraction of the two periodic attractors have a complex *riddled* structure.
Riddled Basins of Attraction

On regions of multistability the basins of attraction of the two periodic attractors have a complex *riddled* structure.
The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that additive white noise in the square root map may be sensible to model systems where the forcing term or external fluctuations represent a significant source of uncertainty.
The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that additive white noise in the square root map may be sensible to model systems where the forcing term or external fluctuations represent a significant source of uncertainty.

The square root map with additive Gaussian white noise is given by

\[ x_{n+1} = S_a(x_n) = \begin{cases} 
\mu + bx_n + \xi_n & \text{if } x_n < 0 \\
\mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0,
\end{cases} \tag{1} \]

where \( \xi_n \) are identically distributed independent normal random variables with mean 0 and standard deviation \( \Delta \), \( \xi_n \sim N(0, \Delta^2) \).
Noisy Bifurcation Diagrams

\[ \begin{align*}
\mu^s_2 & \quad 6.4 & \quad 6.6 & \quad 6.8 & \quad \mu^c_3 \\
-0.04 & \quad -0.02 & \quad 0 & \quad \times 10^{-3}
\end{align*} \]
Noisy Bifurcation Diagrams

\[ x \]
\[ \begin{array}{c}
\mu^s_2 & 6.4 & 6.6 & 6.8 & \mu^e_3 \\
\mu & \times 10^{-3}
\end{array} \]

\[ x \]
\[ \begin{array}{c}
\mu^s_2 & 6.4 & 6.6 & 6.8 & \mu^e_3 \\
\mu & \times 10^{-3}
\end{array} \]
Noisy Bifurcation Diagrams

\begin{center}
\begin{tikzpicture}
  \begin{axis}[
    xmin=5.5e-3, xmax=6.9e-3,
    ymin=-0.04, ymax=0.00,
    xlabel=$\mu$,
    ylabel=$x$,
    xtick={5.5e-3, 6.4e-3, 6.6e-3, 6.8e-3},
    ytick={-0.04, -0.02, 0},
    xticklabels={$\mu_2^s$, 6.4, 6.6, $\mu_3^e$},
    yticklabels={-0.04, -0.02, 0},
    grid=both,
    width=\textwidth,
    height=\textwidth,
    every axis plot/.append style={thick},
  ]
    \addplot[red, only marks] coordinates {(5.5e-3, 0) (6.4e-3, 0) (6.6e-3, 0) (6.8e-3, 0)};
    \addplot[black, only marks] coordinates {(5.5e-3, -0.04) (6.4e-3, 0) (6.6e-3, 0) (6.8e-3, 0)};
  \end{axis}
\end{tikzpicture}
\end{center}

\begin{center}
\begin{tikzpicture}
  \begin{axis}[
    xmin=5.5e-3, xmax=6.9e-3,
    ymin=-0.04, ymax=0.00,
    xlabel=$\mu$,
    ylabel=$x$,
    xtick={5.5e-3, 6.4e-3, 6.6e-3, 6.8e-3},
    ytick={-0.04, -0.02, 0},
    xticklabels={$\mu_2^s$, 6.4, 6.6, $\mu_3^e$},
    yticklabels={-0.04, -0.02, 0},
    grid=both,
    width=\textwidth,
    height=\textwidth,
    every axis plot/.append style={thick},
  ]
    \addplot[red, only marks] coordinates {(5.5e-3, 0) (6.4e-3, 0) (6.6e-3, 0) (6.8e-3, 0)};
    \addplot[black, only marks] coordinates {(5.5e-3, -0.04) (6.4e-3, 0) (6.6e-3, 0) (6.8e-3, 0)};
  \end{axis}
\end{tikzpicture}
\end{center}
Noise Amplitude and Proportions of Periodic Behaviour

Noise Amplitude and Proportions of Periodic Behaviour
Noise Amplitude and Proportions of Periodic Behaviour

Significant Shift Effective Destruction Relationship Reversal

\( \Delta \)

Eoghan Staunton

Advances in Nonsmooth Dynamics

June 2018 7 / 18
Noise Amplitude and Proportions of Periodic Behaviour

Eoghan Staunton

Advances in Nonsmooth Dynamics

June 2018
Threshold Noise Amplitudes

Eoghan Staunton
Basins and Steady State Distributions

Primary basins:

Steady-State $\sigma$s:

We consider threshold values of $\rho = D/\sigma$. $\rho$ gives us some measure of how likely it is for noise to push the dynamics out of the basin of attraction.
Basins and Steady State Distributions

Primary basins:

Steady-State $\sigma$s:

We consider threshold values of $\rho = D/\sigma$. $\rho$ gives us some measure of how likely it is for noise to push the dynamics out of the basin of attraction.
Threshold $\rho$ Values

Significant Shift
Effective Destruction
Relationship Reversal

$\rho$

$\Delta$

$2 \times 10^{-4}$

$\mu$

$\times 10^{-3}$

$\rho$

$5$

$6.4$

$6.6$

$6.8$

$\mu$

$\times 10^{-3}$
Scaling

By investigating the scaling of \( \varrho \) on intervals of multistability of increasing period we can determine how the effect of noise scales.
By investigating the scaling of $\rho$ on intervals of multistability of increasing period we can determine how the effect of noise scales.

We find that choosing the noise amplitude to be $\Delta' = b^2 \Delta$ on the interval of multistability $(\mu_{m+1}^s, \mu_{m+2}^e)$ will result in a similar effect of noise on the dynamics of the map as choosing the noise amplitude to be $\Delta$ on the interval $(\mu_m^s, \mu_{m+1}^e)$ for large $m$. 
Inducing Multistability

We have previously seen that noise of an appropriate amplitude also has the potential to induce multistability in regions close to, but outside, intervals of multistability.
Inducing Multistability

We have previously seen that noise of an appropriate amplitude also has the potential to induce multistability in regions close to, but outside, intervals of multistability.

We have found that noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

\[ RLLRL \ldots RLLRL \overline{RRLRL} \ldots RLRL. \]  \hspace{1cm} (2)
Inducing Multistability

We have previously seen that noise of an appropriate amplitude also has the potential to induce multistability in regions close to, but outside, intervals of multistability.

We have found that noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

\[ RLLRLL \ldots RLLRRLRLRL \ldots RLRL. \] (2)

The significant feature of the symbolic representation of the transition above is the repeated \( R \), corresponding to repeated iteration on the right-hand side of the square root map.
Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

\[ A_{RR} = \left( 0, \left( \mu/a \right)^2 \right) . \]

(3)

We also note that the last left iterate of the period-3 orbit is very close to 0 for values of \( \mu \) close to the interval of multistability.
Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

$$A_{RR} = (0, (\mu/a)^2).$$

We also note that the last left iterate of the period-3 orbit is very close to 0 for values of $\mu$ close to the interval of multistability.

Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into $A_{RR}$, inducing repeated $R$'s or repeated low-velocity impacts.
Noise and Deterministic Structures

\[ RL \quad — \quad RLL \quad — \quad \]

Graphs showing the functions \( T_f \) and \( S^2_f(T_f) \) with x-axis scales of \( 0 \) to \( 1.5 \times 10^{-4} \) and \(-0.04 \) to \( 0.02 \) respectively.
Noise and Deterministic Structures

\[ RL \quad \rightarrow \quad RLL \quad \rightarrow \]

\[ T_f \]

\[ S^2_R(T_f) \]

Iterates to Transition
Noise and Deterministic Structures

\[ RL \quad \rightarrow \quad RLL \quad \rightarrow \]

\begin{align*}
\mathcal{T}_f & \quad (x) \\
S^2_R(T_f) & \quad (x) \\
\text{Iterates to Transition} & \quad (x)
\end{align*}

\[ x \]

\begin{align*}
0 & \quad 500 \\
1000 & \quad 1500 \\
2000 & \quad n
\end{align*}
Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^m RL^m \ldots RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1} \ldots . . . . . . . . RL^{m-1}$$

for $\mu$ in a neighbourhood of $\mu^s_m$ such that $\mu < \mu^s_m$ and $k \in \{2, 3, \ldots, m\}$. 


Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^m RL^m \ldots RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1} \ldots \ldots RL^{m-1}$$

for $\mu$ in a neighbourhood of $\mu_m^s$ such that $\mu < \mu_m^s$ and $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2}R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu_m^s$. 
Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^m RL^m \ldots RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1} \ldots \ldots RL^{m-1}$$

for $\mu$ in a neighbourhood of $\mu_{s_m}$ such that $\mu < \mu_{s_m}$ and $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2}R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu_{s_m}$. 

$\begin{array}{cccc}
A_{RR} & A_{RL^1R} & A_{RL^2R} & A_{RL^3R} \\
\end{array}$
Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^mRL^m ... RL^m RL^{m-1} RL^{k-2} RL^{m-1} RL^{m-1} ... RL^{m-1}$$

for $\mu$ in a neighbourhood of $\mu_m^s$ such that $\mu < \mu_m^s$ and $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2}R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu_m^s$. 
Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^m RL^m \ldots RL^m R^{m-1} R^{k-2} RL^{m-1} RL^{m-1} \ldots \ldots RL^{m-1}$$ (4)

for $\mu$ in a neighbourhood of $\mu^s_m$ such that $\mu < \mu^s_m$ and $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2} R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu^s_m$. 
Conclusions

- Additive noise has a complex nonmonotonic effect on the proportion of iterates spent in coexisting periodic behaviours on intervals of multistability.
- The relationship observed is highly dependent on the value of the bifurcation parameter $\mu$.
- We can explain these relationships by examining how the steady-state distributions associated with periodic orbits interact with their basins of attraction.
- The effect of the addition of noise on intervals of multistability of increasing minimal periodic orbit obeys a scaling law.
- Additive noise has the potential to induce multistability outside such intervals.


______, *Noise induced multistability in the square root map*, Under Review (2018), TBC.