Noise and Multistability in the Square Root Map

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Noise and Nonsmoothness in Dynamical Systems

Both noise and nonsmoothness have been shown to independently be the drivers of significant changes in qualitative behaviour.

- Nonsmooth systems - qualitative changes in the behavior of the system under parameter variation that do not occur in the smooth setting.
- Adding noise to (smooth) systems - does more than just blur the outcome of the system in the absence of noise
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Figure: From [CONG94].

Figure: Adapted from [LL86].
The Square Root Map

Many impacting systems, including rattling gears, moored boats impacting docks, Braille printers, percussive drilling and atomic force microscopes are described by a 1-D map known as the square root map near *grazing* impacts.

\[ x_{n+1} = S(x_n) = \begin{cases} 
\mu + bx_n & \text{if } x_n < 0, \\
\mu - a\sqrt{x_n} & \text{if } x_n \geq 0,
\end{cases} \]

where \( a > 0 \) and \( b > 0 \).

Symbolically, if \( x_n < 0 \) it is represented by an \( L \) and if \( x_n > 0 \) it is represented by an \( R \).
Deriving The Square Root Map

Deriving the map from the full system

Discontinuity surface

Incoming trajectory

Outgoing trajectory

\( D^- \)

\( g_D \)

\( D^+ \)

Jump map

Deriving the map from the full system

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Discontinuity surface

Flow for a time $t_1 < 0$ from until the trajectory intersects $\mathcal{D}^-$

Incoming trajectory

$\mathcal{D}^-$

in

$\mathcal{D}$

$\mathcal{D}^+$

out

Jump map

Outgoing trajectory

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Discontinuity surface

Flow for a time $t_1 < 0$ from until the trajectory intersects $\mathcal{D}^-$

Flow for a time $t_2 < 0$ until the trajectory intersects $\mathcal{P}$

Incoming trajectory

Jump map

Outgoing trajectory

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Discontinuity Mapping

Flow for a time $t_1 < 0$ from until the trajectory intersects $\mathcal{D}^-$

Flow for a time $t_2 < 0$ until the trajectory intersects $\mathcal{P}$

Discontinuity surface

Incoming trajectory

Outgoing trajectory

Jump map

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The Period Adding Cascade

Here we will assume that the parameter $b$ (the slope of the linear part) is such that $0 < b < 1/4$. For values of $b$ in this range the deterministic square root map undergoes a period-adding cascade with intervals of bistability as the bifurcation parameter $\mu$ is decreased.

![Graph showing period adding cascade](image-url)
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These periodic orbits take the form $(RL^m)_{\infty}$ for $m = 1, 2, 3, \ldots$. This means they consist of one iterate on the right ($> 0$) followed by $m$ iterates on the left ($< 0$).
Riddled Basins of Attraction

On regions of bistability the basins of attraction of the two periodic attractors have a complex *riddled* structure.
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The Square Root Map With Additive Noise

In [SHK13] Simpson, Hogan and Kuske show that white noise in the piecewise smooth flow translates to additive white noise in the square root map. This noise formulation may be sensible to model systems where the forcing term or external fluctuations represent a significant source of uncertainty.
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The square root map with additive Gaussian white noise is given by

\[
x_{n+1} = S_a(x_n) = \begin{cases} 
\mu + bx_n + \xi_n & \text{if } x_n < 0 \\
\mu - a\sqrt{x_n} + \xi_n & \text{if } x_n \geq 0,
\end{cases}
\]

where \( \xi_n \) are identically distributed independent normal random variables with mean 0 and standard deviation \( \Delta \), \( \xi_n \sim N(0, \Delta^2) \).
Noisy Bifurcation Diagrams

\[ x \]

\[ \begin{array}{c}
\mu^s_2 & 6.4 & 6.6 & 6.8 & \mu^c_3 \\
0 & & & & \times 10^{-3}
\end{array} \]
Noisy Bifurcation Diagrams
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\[ x \]

\[ \mu \times 10^{-3} \]

\[ \mu_2^s \ 6.4 \ 6.6 \ 6.8 \ \mu_3^c \]

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Noise Amplitude and Proportions of Periodic Behaviour

![Diagram showing the relationship between noise amplitude and proportions of periodic behaviour.](image)

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Noise Amplitude and Proportions of Periodic Behaviour

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Inducing Bistability

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In the numerical simulations we have found that noise-induced transitions from period-3 to period-2 behaviour in regions where period-2 behaviour is unstable display certain similarities. In particular, we have observed that the transitions tend to take the following symbolic form

\[ RLLRL \ldots RLLRLRRLRL \ldots RLRL. \]  

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\[ RLLRLL \ldots RLRRLRRLRL \ldots RLRL. \quad (2) \]

The significant feature of the symbolic representation of the transition above is the repeated \( R \), corresponding to repeated iteration on the right-hand side of the square root map, i.e. repeated low-velocity impacts in the physical system.
Noise and Deterministic Structures

We note that the set of initial values that are on the right which remain on the right after iteration by the deterministic square root map are given by the interval

\[ A_{RR} = (0, (\mu/a)^2) \]  

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We also note that the last left iterate of the period-3 orbit is very close to 0 for values of \( \mu \) close to the interval of multistability.
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We also note that the last left iterate of the period-3 orbit is very close to 0 for values of $\mu$ close to the interval of multistability.

Therefore, it is not hard to see that noise has the potential to push the last left iterate of a period-3 orbit into $A_{RR}$ inducing repeated $R$’s or repeated grazing impacts.
Noise and Deterministic Structures

$RL \quad -$  
$RLL \quad -$
Noise and Deterministic Structures

\[ RL \quad - \quad RLL \quad - \]
Noise and Deterministic Structures

\[ RL \quad \longrightarrow \quad RLL \quad \longrightarrow \]
Noise and Deterministic Structures

$\mathbf{RL}$

$\mathbf{RLL}$

\[ \times 10^4 \]

\[ \times 10^{-4} \]

$T_f$

$x$

$S^2_R(T_f)$

$x$

Iterates to Transition

$\mathbf{x}$

$\mathbf{n}$

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Generalising to Higher Periodicities

The features of this transition are repeated as we look at transitions from $RL^m$ behaviour to $RL^{m-1}$ behaviour for increasing $m$. In particular we observe transitions of the form

$$RL^m RL^m ... RL^m R L^{m-1} RL^{k-2} R L^{m-1} R L^{m-1} ... R L^{m-1}$$

(4)

for $\mu$ in a neighbourhood of $\mu^s_m$ such that $\mu < \mu^s_m$ and $k \in \{2, 3, \ldots, m\}$. 

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for $\mu$ in a neighbourhood of $\mu_m^s$ such that $\mu < \mu_m^s$ and $k \in \{2, 3, \ldots, m\}$. The most significant feature of this transition is the sequence $RL^{k-2} R$ for $k \in \{2, 3, \ldots, m\}$, corresponding to iterations on the right-hand side of the map being repeated more quickly than is usual for a settled system with $\mu < \mu_m^s$. 
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\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{chart.png}
\caption{Diagram of the system with different regions for $A_{RR}, A_{RL1R}, A_{RL2R}, A_{RL3R}$.
\end{figure}
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![Graph showing iterations on the right-hand side of the map being repeated more quickly](image)
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![Image of bar chart and graphs]
Conclusions

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- Repeated low-velocity impacts play an important role in noise-induced transitions from stable to unstable periodic behaviour.
- This behaviour can be generalised to higher periodicities.
- The effect of the addition of noise on intervals of multistability of increasing minimal periodic orbit obeys a scaling law.


______, *Noise induced multistability in the square root map*, Under Review (2018), TBC.