Alternating Sign Matrices and Their Corresponding Bipartite Graphs

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The number of $n \times n$ *ASMs* is $\frac{1!4!7!\ldots(3n-2)!}{n!(n+1)!(n+2)!\ldots(2n-1)!}$.

Examples

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

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Alternating Signed Bipartite Graphs

Associated to each ASM is an *alternating signed bipartite graph*. This graph has a vertex for each row and column of the matrix. Vertex r_i is connected to vertex c_j by a positive edge (represented in blue) if there is a 1 in position (i, j) of the matrix, and by a negative edge (represented in red) if there is a -1 in position (i, j).

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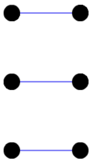

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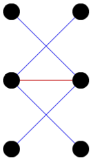
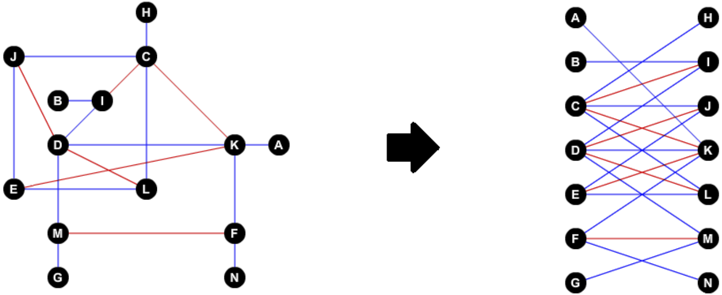


Identifying ASBGs



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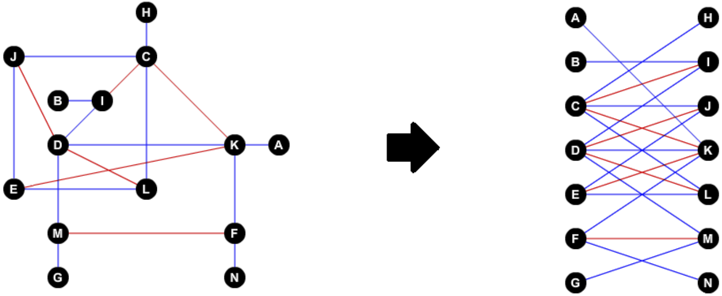

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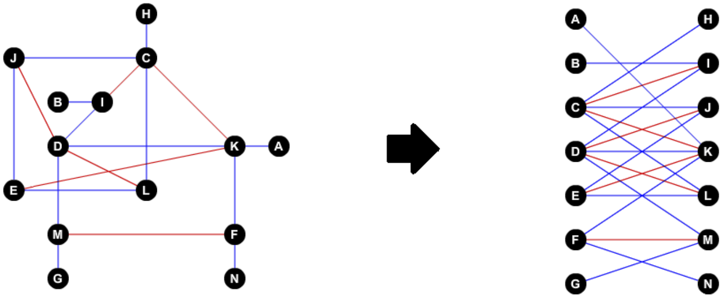

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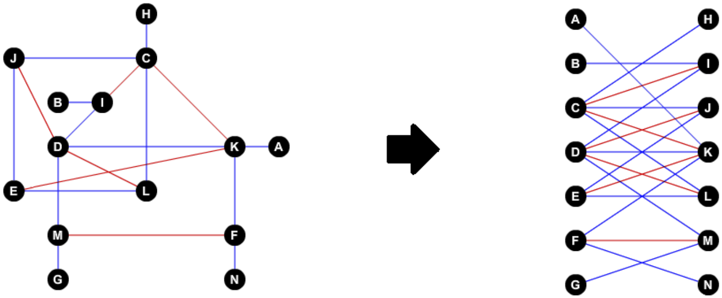

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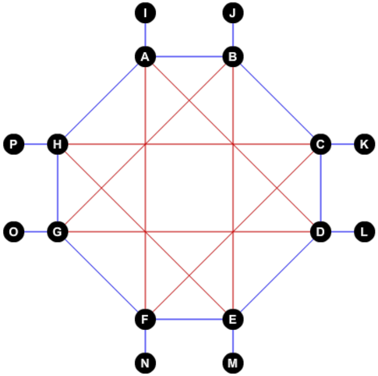
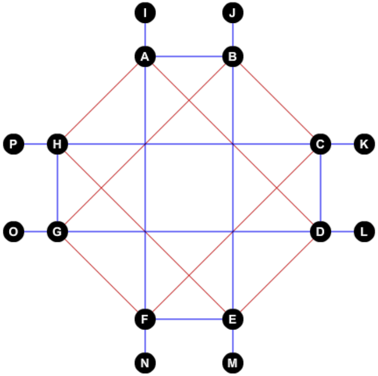
- The graph must be bipartite.
- The graph must be *balanced*.
- $deg_b(v_i) = deg_r(v_i) + 1, \forall i = 1, 2, \ldots, 2n.$

Motivating Counter Example





ASBG Core

When trying to determine if a graph is isomorphic to an ASBG, it is useful to define the *core* of a graph. The core of a graph is the subgraph that remains after removing leaves from the graph. There are restrictions on what the core of an ASBG can be:

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- The blue core must be bipartite, and it must be possible to embed it in a plane in bipartite form so that no vertex is connected to two vertices that are consecutive in the embedding.

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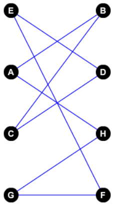

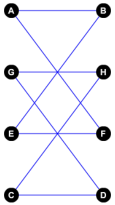

Separation Graphs

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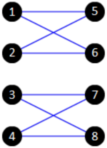

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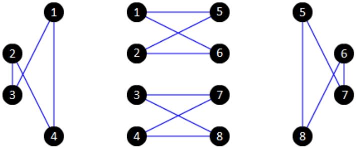

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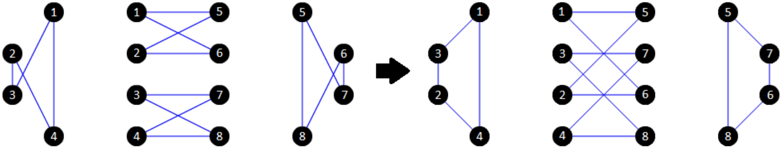

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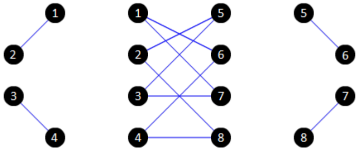

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