Counting Matrices with a Given Rank and Stable Rank over $M_n(\mathbb{F}_p)$

Cian O'Brien   Supervisor: Rachel Quinlan

Summer Internship

October 7th, 2016

Introduction

- This project was concerned with studying the properties of matrices over a finite field.

Introduction

- This project was concerned with studying the properties of matrices over a finite field.
- The main theme is an extension and reinterpretation of a method that was presented by Crabb (2003) for counting the number of nilpotent $n \times n$ matrices over a finite field with specified order.

Introduction

- This project was concerned with studying the properties of matrices over a finite field.
- The main theme is an extension and reinterpretation of a method that was presented by Crabb (2003) for counting the number of nilpotent $n \times n$ matrices over a finite field with specified order.
- Crabb establishes a bijective correspondence between the set of nilpotent matrices in $M_n(\mathbb{F}_p)$ and the set of sequences of length $n-1$ of vectors of length n over $\mathbb{F}_p$. He used this to deduces that the number of nilpotent matrices is $p^{n(n-1)}$.

Introduction

- This project was concerned with studying the properties of matrices over a finite field.
- The main theme is an extension and reinterpretation of a method that was presented by Crabb (2003) for counting the number of nilpotent $n \times n$ matrices over a finite field with specified order.
- Crabb establishes a bijective correspondence between the set of nilpotent matrices in $M_n(\mathbb{F}_p)$ and the set of sequences of length $n-1$ of vectors of length n over $\mathbb{F}_p$. He used this to deduces that the number of nilpotent matrices is $p^{n(n-1)}$.
- We generalise Crabb's construction to a bijection involving all elements of $M_n(\mathbb{F}_p)$ and show how it can be used to count the numbers of matrices with specified stable rank and rank power sequence.

We begin with some definitions:

- The *rank* of an $n \times n$ matrix M is the number of linearly independent rows or columns of the matrix.

We begin with some definitions:

- The *rank* of an $n \times n$ matrix M is the number of linearly independent rows or columns of the matrix.
- The *stable rank* of an $n \times n$ matrix M is the rank of M^n.

We begin with some definitions:

- The *rank* of an $n \times n$ matrix M is the number of linearly independent rows or columns of the matrix.
- The *stable rank* of an $n \times n$ matrix M is the rank of M^n.
- The *index* of an $n \times n$ matrix M is the lowest natural number k for which $\text{rank}(M^k) = \text{rank}(M^n)$.

We begin with some definitions:

- The *rank* of an $n \times n$ matrix M is the number of linearly independent rows or columns of the matrix.
- The *stable rank* of an $n \times n$ matrix M is the rank of M^n.
- The *index* of an $n \times n$ matrix M is the lowest natural number k for which $\text{rank}(M^k) = \text{rank}(M^n)$.
- An $n \times n$ matrix M is *nilpotent* if some power of M is the zero matrix.

We begin with some definitions:

- The *rank* of an $n \times n$ matrix M is the number of linearly independent rows or columns of the matrix.
- The *stable rank* of an $n \times n$ matrix M is the rank of M^n.
- The *index* of an $n \times n$ matrix M is the lowest natural number k for which $\text{rank}(M^k) = \text{rank}(M^n)$.
- An $n \times n$ matrix M is *nilpotent* if some power of M is the zero matrix.

  Note: A matrix is nilpotent if its stable rank is zero

Examples

$$A = \begin{pmatrix} 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is nilpotent, as } A^4 = 0$$

Examples

$$A = \begin{pmatrix} 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is nilpotent, as } A^4 = 0$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is not nilpotent}$$

Examples

$$A = \begin{pmatrix} 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ is nilpotent, as } A^4 = 0$$

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{is not nilpotent}$$

$$B^3 = B^4 = ... = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{so B has stable rank 1}$$

Crabb's Algorithm

Let M be an $n \times n$ nilpotent matrix of index k. Let V_0, V_1, ... , V_k be ordered bases of the column spaces of M^0, M^1, ... ,M^k.Let V = V_0 and "adapt" the basis V as follows. Starting with i = 1:

- Let W_i be the matrix that has the jth vector of V_i as its jth row, over basis V. Reduce this to row echelon form.
- Let c be the number of columns in W_i, and p be the number of pivots. Let the first $c - p$ elements of the adapted basis V' be the jth element of V, where j takes the values of the columns without pivots. Then let the remaining elements be the sum of the jth entry in a row multiplied by the jth entry in V, for $j = 1, 2, ..., n$
- Then let $V = V'$, and repeat the process for the next value of i, until $i = k$.

Now V has been fully adapted. Finally, multiply each vector in V on the left by M, and these are the entries in the tuple.

Extending the Algorithm

We extended this algorithm to apply to all $n \times n$ matrices. In the extended algorithm, each $n \times n$ matrix has a corresponding n-tuple, instead of an $(n-1)$-tuple. We were then able to use this tuple to look at interesting properties of matrices, over a finite field of order p. For example, we were able to derive a formula to count the number of $n \times n$ matrices over a field of order p, with rank r and stable rank s. In order to do this, we needed to define a few more terms:

Extending the Algorithm

We extended this algorithm to apply to all $n \times n$ matrices. In the extended algorithm, each $n \times n$ matrix has a corresponding n-tuple, instead of an $(n-1)$-tuple. We were then able to use this tuple to look at interesting properties of matrices, over a finite field of order p. For example, we were able to derive a formula to count the number of $n \times n$ matrices over a field of order p, with rank r and stable rank s. In order to do this, we needed to define a few more terms:

- The *rank array* of an $n \times n$ matrix M is a non-decreasing sequence of integers. The kth value of the *rank array* is the dimension of the span of the last k entries in the tuple corresponding to M.

Extending the Algorithm

We extended this algorithm to apply to all $n \times n$ matrices. In the extended algorithm, each $n \times n$ matrix has a corresponding n-tuple, instead of an $(n-1)$-tuple. We were then able to use this tuple to look at interesting properties of matrices, over a finite field of order p. For example, we were able to derive a formula to count the number of $n \times n$ matrices over a field of order p, with rank r and stable rank s. In order to do this, we needed to define a few more terms:

- The *rank array* of an $n \times n$ matrix M is a non-decreasing sequence of integers. The kth value of the *rank array* is the dimension of the span of the last k entries in the tuple corresponding to M.
- The *power sequence* of an $n \times n$ matrix M is a strictly increasing sequence of integers [rank(M^k), rank(M^{k-1}), ..., rank(M^2), rank(M), rank(I_n)], where k is the index of M.

Proposition

While examining this extended algorithm, we discovered the following:

An $n \times n$ matrix M is *nilpotent* if the first entry in the *rank array* is zero. Otherwise, the *stable* rank is the first repeated entry in the *rank array*.

Formulae

This led us to the derivation of two formulae:

Formulae

This led us to the derivation of two formulae:

- The number M_1 of $n \times n$ matrices over the finite field of order p with rank r, and stable rank s is as follows:

$$\mathbf{M_1(n,p,r,s) = p^{(n-r)s}\binom{n-s-1}{r-s}_p \prod_{i=0}^{r-1}(p^n - p^i)}$$

Formulae

This led us to the derivation of two formulae:

- The number M_1 of $n \times n$ matrices over the finite field of order p with rank r, and stable rank s is as follows:

$$\mathbf{M_1(n,p,r,s) = p^{(n-r)s} \binom{n-s-1}{r-s}_p \prod_{i=0}^{r-1} (p^n - p^i)}$$

- The number M_2 of $n \times n$ matrices over the finite field of order p with rank r, stable rank s, and power sequence $L = [s = l_1,\ l_2,\ ...,\ l_{k-1} = r,\ l_k\ = n]$ is as follows:

$$\mathbf{M_2(n,p,r,s,L) = p^s \prod_{i=0}^{r-1} (p^n - p^i) \prod_{j=1}^{k-2} p^{l_{j+2} - 2l_{j+1} + l_j} \binom{l_{j+2} - l_{j+1}}{l_{j+1} - l_j}_p}$$

Crabb, M.C., *Counting nilpotent endomorphisms*, Finite Fields and Their Applications 12(151-154), 2005

Weisstein, Eric W., *q-Binomial Coefficient*, http://mathworld.wolfram.com/q-BinomialCoefficient.html