Naturality and Bockstein homomorphism

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Overview

- Chain and cochain complex
- homology and cohomology
- connecting homomorphism
- Bockstein homomorphism

Chain and cochain complex

- Let R be a ring. A *chain complex* $C = (C_n, d_n)_{n\in\mathbb{Z}}$ of R-modules is a sequence of homomorphisms of R-modules

$$\cdots \longrightarrow C_{n+1} \xrightarrow{d_{n+1}} C_n \xrightarrow{d_n} C_{n-1} \xrightarrow{d_{n-1}} \cdots \tag{1}$$

such that $d_n d_{n+1} = 0$ for all n. The *chain complex* C *is called exact sequence if Im* d_{n+1}= *Ker* d_n *for all* n*. The operation* d_n *are boundary maps.*

Chain and cochain complex

- Dualize the chain complex 1, that is means apply $Hom(-, G)$ to it. to get the cochain complex.

$$\cdots \overset{\delta^{n+1}}{\longleftarrow} C^{n+1} \overset{\delta^{n}}{\longleftarrow} C^{n} \overset{\delta^{n-1}}{\longleftarrow} C^{n-1} \longleftarrow \cdots \tag{2}$$

with $C^n := Hom(C_n, G)$, and the coboundary map $\delta^n : C^n \longrightarrow C^{n+1}$ is define by

$$(\delta^n \psi)(\alpha) = \psi(d_{n+1}\alpha), \ \textit{for } \psi \in C^n \textit{ and } \ \alpha \in C_{n+1}. \tag{3}$$

with the property that $\delta^n \circ \delta^{n+1} = 0$.

Homology and cohomology

Definition

Let $C = (C_n, d_n)_{n\in\mathbb{Z}}$ be a *chain complex* of R-modules and let $C = (C^n, \delta^n)_{n\in\mathbb{Z}}$ be a *cochain complex* of R-modules . For each $n \in \mathbb{Z}$, the *nth (co)homology module* of C is defined to be the quotient module

$$H_n(C) = \frac{ker d_n}{Im d_{n+1}}$$

$$H^n(C) = \frac{ker \delta^n}{Im \delta^{n-1}}$$

Resolution

- Let G be a group. A *free* $\mathbb{Z}G$*-resolution* of $\mathbb{Z}$ is an exact sequence of $\mathbb{Z}G$-modules

$$R_*^G : \cdots \longrightarrow R_{n+1}^G \xrightarrow{\partial_{n+1}} R_n^G \xrightarrow{\partial_n} R_{n-1}^G \longrightarrow \cdots \longrightarrow R_1^G \xrightarrow{\partial_1} R_0^G$$

with each R_i^G a free $\mathbb{Z}G$-module for all $i \geq 0$.

- Given R_*^G a -resolution of and any A be a module we can use the tensor product to constuct an induced chain complex $R_*^G \otimes_{\mathbb{Z}G} A$ of abelian groups

$$\cdots \longrightarrow R_{n+1}^G \otimes_{\mathbb{Z}G} A \xrightarrow{d_{n+1}} R_n^G \otimes_{\mathbb{Z}G} A \xrightarrow{d_n} R_{n-1}^G \otimes_{\mathbb{Z}G} A \longrightarrow \cdots$$

Resolution

- We can also construct an induced cochain complex $Hom(R_*^G, A)$ of abelian groups

$$\cdots \longrightarrow Hom_{\mathbb{Z}G}(R_{n-1}, A) \xrightarrow{d^{n-1}} Hom_{\mathbb{Z}G}(R_n, A) \xrightarrow{d^n} Hom_{\mathbb{Z}G}(R_{n+1}, A)$$

- Let R_*^G a *free* $\mathbb{Z}G$*-resolution* of $\mathbb{Z}$ and A be a $\mathbb{Z}G$-module. The *homology* and cohomology of a group G with coefficients in a -module A are defined to be the -modules

$$H_n(G, A) = H_n(R_*^G \otimes_{\mathbb{Z}G} A),$$

$$H^n(G, A) = H^n(Hom(R_*^G, A)), \ \textit{for all} \ n \geq 0.$$

connecting homomorphism

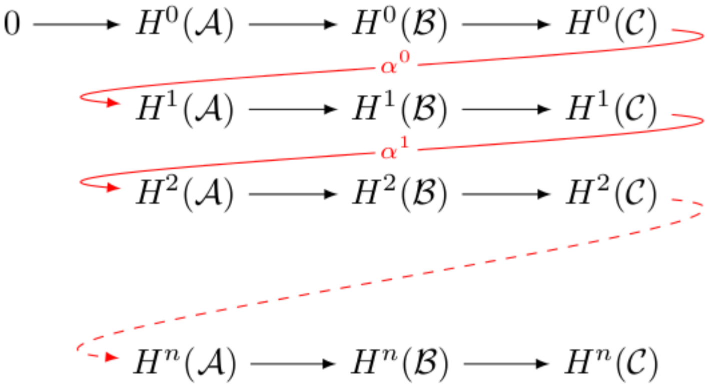
Proposition

[1] Let $0 \longrightarrow K \xrightarrow{i} A \xrightarrow{\phi} B \longrightarrow 0$ *be a short exact sequence of* $\mathbb{Z}G$*-modules. Then there is a (natural) connecting homomorphism*

$$\delta^n : H^n(G, B) \longrightarrow H^{n+1}(G, K),$$

such that the following sequence is exact:

$$\cdots H^n(G, K) \xrightarrow{i_*} H^n(G, A) \xrightarrow{p_*} H^n(G, B) \xrightarrow{\delta^n} H^{n+1}(G, K) \longrightarrow \cdots$$


0
$H^0(\mathcal{A})$
$H^0(\mathcal{B})$
$H^0(\mathcal{C})$
α^0
$H^1(\mathcal{A})$
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α^1
$H^2(\mathcal{A})$
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$H^2(\mathcal{C})$
$H^n(\mathcal{A})$
$H^n(\mathcal{B})$
$H^n(\mathcal{C})$

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
H^{n+1}(G,K) & & CK_{n+2} & \xrightarrow{i^*_{n+2}} & CA_{n+2} & \xrightarrow{\phi_{n+2}} & CB_{n+2} & & \\
\uparrow{\scriptstyle \eta_K^{n+1}} & & \uparrow{\scriptstyle d_K^{n+1}} & & \uparrow{\scriptstyle d_A^{n+1}} & & \uparrow{\scriptstyle d_B^{n+1}} & & \\
ker\, d_K^{n+1} & \xrightarrow{incl.} & CK_{n+1} & \xrightarrow{i^*_{n+1}} & CA_{n+1} & \xrightarrow{\phi_{n+1}} & CB_{n+1} & & H^n(G,B) \\
\uparrow{\scriptstyle i} & & \uparrow{\scriptstyle d_K^{n}} & & \uparrow{\scriptstyle d_A^{n}} & & \uparrow{\scriptstyle d_B^{n}} & & \uparrow{\scriptstyle \eta_B^{n}} \\
Im_K^n & & CK_n & \xrightarrow{i^*_{n}} & CA_n & \xrightarrow{\phi_n} & CB_n & \xleftarrow{incl.} & ker\, d_B^n \\
 & & & & \uparrow{\scriptstyle d_A^{n-1}} & & \uparrow{\scriptstyle d_B^{n-1}} & & \uparrow{\scriptstyle i} \\
 & & & & CA_{n-1} & \xrightarrow{\phi_{n-1}} & CB_{n-1} & & Im\, d_B^{n-1} \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \vdots & & \vdots & &
\end{array}
$$

Example

Consider the symmetric group $G = S_4$ and the sequence $\mathbb{Z}/4\mathbb{Z} \longrightarrow \mathbb{Z}/8\mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z}$ of trivial $\mathbb{Z}G$-modules. We can represent a $\mathbb{Z}G$-module as a G-`outer` group. The following commands use this representation to compute the induced cohomology homomorphism $f : H^3(S_4, \mathbb{Z}/4\mathbb{Z}) \longrightarrow H^3(S_4, \mathbb{Z}/8\mathbb{Z})$ and determine that its kernel has order 2.

GAP session

$gap > G := SymmetricGroup(4);;$
$gap > x := (1, 2, 3, 4, 5, 6, 7, 8);;$
$gap > a := Group(x^2);;$
$gap > b := Group(x);;$
$gap > ahomb := GroupHomomorphismByFunction(a, b, y- > y);;$
$gap > phi!.Mapping := ahomb;;$
$gap > HA := CohomologyHomomorphism(phi, 3).$
$gap > Size(KernelOfGOuterGroupHomomorphism(HA));$
2

We have implemented the algorithm of the diagram above as a function ConnectingCohomologyHomomorphism(psi,n) in the HAP .

Example

The following commands then compute the connecting homomorphism

$$H^2(S_4, \mathbb{Z}/2\mathbb{Z}) \longrightarrow H^3(S_4, \mathbb{Z}/4\mathbb{Z})$$

and determine that the image of this homomorphism has order 2.

GAP session

```
gap > C := TrivialGModuleAsGOuterGroup(G, Image(bhomc));
gap > psi := GOuterGroupHomomorphism();
gap > psi!.Source := B;
gap > psi!.Target := C;
gap > psi!.Mapping := bhomc;
gap > delta := ConnectingCohomologyHomomorphism(psi, 2);;
gap > Size(ImageOfGOuterGroupHomomorphism(delta));
2
```

Bockstein homomorphism

Definition

The Bockstein homomorphism is the connecting homomorphism in the exact sequence

$$\cdots \longrightarrow H^n(B,\mathbb{Z}_p) \xrightarrow{i_*} H^n(B,\mathbb{Z}_{p^2}) \xrightarrow{p_*} H^n(B,\mathbb{Z}_p) \xrightarrow{\beta} H^{n+1}(B,\mathbb{Z}_p) \longrightarrow$$

obtained by applying Proposition 0.1 to the exact sequence of coefficient modules

$$\mathbb{Z}_p \rightarrowtail \mathbb{Z}_{p^2} \twoheadrightarrow \mathbb{Z}_p.$$

Proposition

Let $x, y \in H^*(B, \mathbb{Z}_p)$ *we have*

$$\beta(x \smile y) = \beta(x) \smile y + (-1)^{|x|} x \smile \beta(y) \tag{4}$$

and

$$\beta(\beta(x)) = 0 \tag{5}$$

In light of 5 we can define the Bockstein cohomology

$$BH^n(B, \mathbb{Z}_p) = \frac{Ker(\beta : H^n(B, \mathbb{Z}_p) \longrightarrow H^{n+1}(B, \mathbb{Z}_p))}{Im(\beta : H^{n-1}(B, \mathbb{Z}_p) \longrightarrow H^n(B, \mathbb{Z}_p))}. \tag{6}$$

The definition of β and $BH^n(B, \mathbb{Z}_p)$ are readily implemented on a computer

Example

The following GAP session computes $BH^n(G_{32,15}, \mathbb{Z}_2) = \mathbb{Z}_2$ for all n for $0 \leq n \leq 14$ where $G_{32,15}$ is the small group(32,15) of order 32 in the GAP's library.

GAP session

$gap > G := SmallGroup(32, 15);;$
$gap > A := ModPSteenrodAlgebra(G, 15);;$
$gap > List([1..14], n- > BocksteinHomology(A, n));$
$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
$gap > gens := ModPRingGenerators(A);$
$[v.1, v.2, v.3, v.4, v.6, v.10]$
$gap > List(gens, A!.degree);$
$[0, 1, 1, 2, 3, 4]$
$gap > List(gens, x- > Bockstein(A, x));$
$[0 * v.1, 0 * v.1, v.5, 0 * v.1, v.8, v.11 + v.12]$

- Brown, Kenneth S. *Cohomology of Groups*. Graduate Texts in Mathematics, Springer, 1994.
- Graham Ellis. *HAP-Homological Algebra Programming, Version 1.11.3*.
- R.E. Mosher and M.C. Tangora. *Cohomology Operations and Applications in Homotopy Theory*. Harper and Row, Publishers, New York, 1968.