Postgraduate Modelling Research Group, 17 Feb 2017

Maximum Principles in Differential Equations

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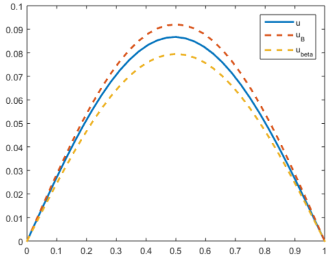


§Outline

For more, see [1].

The Maximum Principle

Suppose a function u that is continuous on $[0, 1]$ takes on its maximum at a point on this interval. If u has a continuous second derivative, and has local maximum at some point c between 0 and 1, then

$$u'(c) = 0 \text{ and } \quad u''(c) \leqslant 0.$$

Suppose that in an open interval $(0, 1)$, u is known to satisfy a differential inequality of the form

$$L(u) \equiv u'' + K(x)u' > 0,$$

where $K(x)$ is any bounded function. The maximum of u in the interval can not be attained anywhere except at the endpoints, 0 or 1. We have here the simplest example of maximum principle.

If u is the solution to

$$-u''(x) + b(x)u(x) = f(x) \quad \text{with } u(0) = u(1) = 0 \text{ on } (0, 1)$$

where b is some function such that $b(x) \geqslant \beta > 0$, β is constant.

Then, we can prove $u(x) \geqslant 0$ for all x.

Proof:

We now have a lower bound for $u(x)$, $(u \geqslant 0)$.

- Can we find an upper bound?
- Yes! Using the maximum principal, we can show that $u(x) \leqslant \|f\|/\beta$.

Proof:

Suppose $\beta \leqslant b(x) \leqslant B$, where B is constant.

Let u_B be the solution to the constant coefficient ODE:

$$L_B u_B := -u_B'' + B u_B = f \qquad \text{with} \qquad u_B(0) = u_B(1) = 0.$$

Let $w = u - u_B$. Then,

$$L_B w(x) = L_B u(x) - L_B u_B(x) = \big(-u''(x) + Bu(x)\big) - f(x)$$

$$\geqslant \underbrace{\big(-u'' + b(x)u\big)}_{f(x)} - f(x) = 0$$

because $B \geqslant b(x)$ and $u \geqslant 0$. So

$$L_B w \geqslant 0 \text{ and thus } w \geqslant 0.$$

It follows that $u(x) \geqslant u_B(x)$.

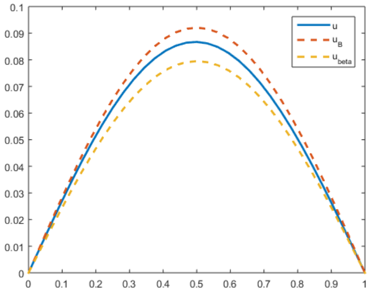
Similarly, let u_β *solve*

$$L_\beta u_\beta := -u_\beta'' + b u_\beta = f$$

So

$$u_B \leqslant u(x) \leqslant u_\beta.$$

So we can bound u *above and below by solutions to constant coefficient equations.*



Conclusion

There are many other applications and generalisations of Maximum Principles:

* the extend to time-dependent problems, and elliptic PDEs;

* they can be used to show that the solution to a PDE shares the qualitative properties of the phenomenon it models (e.g., where a negative solution makes no physical sense);

* there are versions that apply to finite difference equations, known as "Discrete Maximum Principles", and with can be used to analyse finite difference methods.

* in that context, they are related to M-matrices in linear algebra.

References

Murray H. Protter and Hans F. Weinberger.
Maximum principles in differential equations.
Springer-Verlag, New York, 1984.
Corrected reprint of the 1967 original.