Maximum Principles in Differential Equations

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Outline

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For more, see [1].
The Maximum Principle

Suppose a function $u$ that is continuous on $[0, 1]$ takes on its maximum at a point on this interval. If $u$ has a continuous second derivative, and has local maximum at some point $c$ between 0 and 1, then

$$u'(c) = 0 \text{ and } u''(c) \leq 0.$$ 

Suppose that in an open interval $(0, 1)$, $u$ is known to satisfy a differential inequality of the form

$$L(u) \equiv u'' + K(x)u' > 0,$$

where $K(x)$ is any bounded function. The maximum of $u$ in the interval cannot be attained anywhere except at the endpoints, 0 or 1. We have here the simplest example of maximum principle.
Maximum Principles for ODEs

If $u$ is the solution to

$$-u''(x) + b(x)u(x) = f(x) \quad \text{with } u(0) = u(1) = 0 \text{ on } (0, 1)$$

where $b$ is some function such that $b(x) \geq \beta > 0$, $\beta$ is constant. Then, we can prove $u(x) \geq 0$ for all $x$.

Proof:
We now have a lower bound for \( u(x) \), \( u \geq 0 \).

- Can we find an upper bound?
- Yes! Using the maximum principal, we can show that \( u(x) \leq \|f\|/\beta \).

**Proof:**
Suppose $\beta \leq b(x) \leq B$, where $B$ is constant.

Let $u_B$ be the solution to the constant coefficient ODE:

$$L_B u_B := -u_B'' + B u_B = f \quad \text{with} \quad u_B(0) = u_B(1) = 0.$$ 

Let $w = u - u_B$. Then,

$$L_B w(x) = L_B u(x) - L_B u_B(x) = ( -u''(x) + B u(x)) - f(x)$$

$$\geq \left( -u'' + b(x)u \right) - f(x) = 0$$

because $B \geq b(x)$ and $u \geq 0$. So

$$L_B w \geq 0 \text{ and thus } w \geq 0.$$ 

It follows that $u(x) \geq u_B(x)$. 
Maximum Principles for ODEs

Similarly, let \( u_\beta \) solve

\[
L_\beta u_\beta := -u_\beta'' + bu_\beta = f
\]

So

\[
u_B \leq u(x) \leq u_\beta.
\]

So we can bound \( u \) above and below by solutions to constant coefficient equations.
Conclusion

There are many other applications and generalisations of Maximum Principles:
* the extend to time-dependent problems, and elliptic PDEs;
* they can be used to show that the solution to a PDE shares the qualitative properties of the phenomenon it models (e.g., where a negative solution makes no physical sense);
* there are versions that apply to finite difference equations, known as ”Discrete Maximum Principles”, and with can be used to analyse finite difference methods.
* in that context, they are related to M-matrices in linear algebra.
Murray H. Protter and Hans F. Weinberger.

*Maximum principles in differential equations.*
Corrected reprint of the 1967 original.