Equibiaxial Deformations in Dielectric Elastomers

Hannah Conroy Broderick

National University of Ireland, Galway

10 March 2017

Introduction

Electroelasticity is used to model materials, such as electro-active polymers, that deform elastically in an electric field.

In order to model the interaction between electrostatics and non-linear elasticity, we focus on a deformable and electrically polarizable material.

An **equibiaxial** deformation is one where the material is deformed equally along two axes.

Introduction

Schematic of equibiaxial mechanical loading from experiments conducted by Suo group in Harvard.

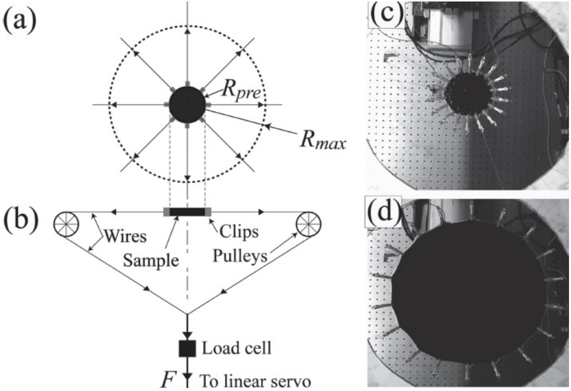

Figure: Experimental setup (from [4])

Equibiaxial Deformation

In an incompressible rectangular slab, an equibiaxial deformation gives principal stretches of the form $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \lambda^{-2}$. In this case, the deformation gradient $\mathbf{F}$ is a diagonal matrix with entries equal to the principal stretches.

Since $\mathbf{F}$ is symmetric, $\mathbf{b} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$ and $\mathbf{c} = \mathbf{F}^{\mathrm{T}}\mathbf{F}$, the left and right Cauchy-Green deformation tensors are equal.

Electroelastic Deformation

Using the theory laid out by Dorfmann and Ogden [1], we model the dielectric using the total Cauchy stress tensor, $\boldsymbol{\tau}$, and the Lagrangian formulation.

$$\mathbf{E}_{\mathrm{L}} = \mathbf{F}^{\mathrm{T}}\mathbf{E}, \qquad \mathbf{D}_{\mathrm{L}} = \mathbf{F}^{-1}\mathbf{D} \tag{1}$$

Depending on the problem we wish to model, we choose either the electric field or the electric displacement as our independent electric variable. We define the total energy density in one of two ways,

$$\Omega = \Omega(\mathbf{F}, \mathbf{E}_{\mathrm{L}}) \quad \text{or} \quad \Omega^{*} = \Omega^{*}(\mathbf{F}, \mathbf{D}_{\mathrm{L}}) \tag{2}$$

Constitutive Equations

Using the definition based on the electric displacement, Ω^*, we get the following constitutive equations for the stress and electric field of an incompressible material,

$$\boldsymbol{\tau} = \mathbf{F}\frac{\partial \Omega^*}{\partial \mathbf{F}} - p^*\mathbf{I}, \qquad \mathbf{E} = \mathbf{F}^{-\mathrm{T}}\frac{\partial \Omega^*}{\partial \mathbf{D}_{\mathrm{L}}} \tag{3}$$

If the material is isotropic, Ω^* depends on 6 invariants, the three principal invariants of $\mathbf{c}$ and three independent invariants that depend on $\mathbf{D}_{\mathrm{L}}$. One possible choice is the following

$$I_1 = \mathrm{tr}(\mathbf{c}), \qquad I_2 = \frac{1}{2}[I_1^2 - \mathrm{tr}(\mathbf{c}^2)], \qquad I_3 = \det(\mathbf{c}),$$
$$I_4 = \mathbf{D}_{\mathrm{L}} \cdot \mathbf{D}_{\mathrm{L}}, \qquad I_5 = (\mathbf{c}\mathbf{D}_{\mathrm{L}}) \cdot \mathbf{D}_{\mathrm{L}}, \qquad I_6 = (\mathbf{c}^2\mathbf{D}_{\mathrm{L}}) \cdot \mathbf{D}_{\mathrm{L}}$$

Constitutive Equations

Using these invariants and the equations (3) for $\boldsymbol{\tau}$ and $\mathbf{E}$, we get the following,

$$\boldsymbol{\tau} = 2\Omega_1^*\mathbf{b} + 2\Omega_2^*(I_1\mathbf{b} - \mathbf{b}^2) - p^*\mathbf{I} \\ + 2\Omega_5^*\mathbf{D}\otimes\mathbf{D} + 2\Omega_6^*(\mathbf{D}\otimes\mathbf{bD} + \mathbf{bD}\otimes\mathbf{D})$$

$$\mathbf{E} = 2(\Omega_4^*\mathbf{b}^{-1} + \Omega_5^*\mathbf{I} + \Omega_6^*\mathbf{b})\mathbf{D} \tag{4}$$

where $\Omega_k^* = \partial\Omega^*/\partial I_k$ for $k = 1, 2, ...6$.
These equations are the constitutive relations for the stress and electric field in an incompressible isotropic electroelastic material.

Equibiaxial Case

We consider an incompressible rectangular slab under an equibiaxial deformation. We apply a voltage to the thickness direction of the slab, so that the electric displacement is given by $\mathbf{D} = (0, 0, D_3)$.

Since the material is incompressible, $I_3 = \det(\mathbf{c}) = 1$, so Ω^* only depends on the other 5 invariants. For this deformation,

$$I_1 = 2\lambda^2 + \lambda^{-4}, \qquad I_2 = \lambda^4 + 2\lambda^{-2},$$
$$I_4 = D_L^2, \qquad I_5 = \lambda^{-4} D_L^2, \qquad I_6 = \lambda^{-8} D_L^2$$

where $D_L = \lambda^2 D_3$ is the magnitude of $\mathbf{D}_\mathrm{L} = (0, 0, D_L)$.

Equibiaxial Case

Since the invariants depend only on λ and D_L, we can introduce a new function $\omega^* = \omega^*(\lambda, D_L)$ which depends only on these two variables.

Using this new function, the invariants and the equations (4) for $\boldsymbol{\tau}$ and $\mathbf{E}$, we get simplified versions of the constitutive equations.

$$\tau_{11} = \tau_{22} = \frac{1}{2}\lambda\omega^*_\lambda$$
$$E_3 = \lambda^2\omega^*_{D_L}$$

where ω^*_λ is the derivative of ω^* with respect to λ, $\omega^*_{D_L}$ is the derivative with respect to D_L, and all other entries are equal to zero.

Equibiaxial Case

These equations can then be transformed to the Lagrangian configuration so that,

$$s = \lambda^{-1}\boldsymbol{\tau}_{11} = \frac{1}{2}\omega^*_\lambda, \qquad E_L = \omega^*_{D_L} \tag{5}$$

We can then calculate the stress and electric field in a material undergoing an equibiaxial deformation for different energy density functions. For example the following function from Suo [2],

$$\Omega^* = \frac{\mu}{2}(I_1 - 3) + \frac{D_L^2}{2\varepsilon}\lambda_3^2 \tag{6}$$

Example

$$\omega^*(\lambda, D_L) = \frac{\mu}{2}(2\lambda^2 + \lambda^{-4} - 3) + \frac{D_L^2}{2\varepsilon}\lambda^{-4} \tag{7}$$

This equation gives the following,

$$s = \mu(\lambda - \lambda^{-5}) - \frac{D_L^2}{\varepsilon}\lambda^{-5}$$

$$E_L = \frac{D_L}{\varepsilon}\lambda^{-4}$$

Using these equations, we can calculate a dimensionless measure of the voltage ($\bar{V} = \sqrt{\frac{\varepsilon}{\mu}}E_L$) in terms of the stretch λ and s/μ.

$$\bar{V} = \sqrt{\lambda^{-2} - \lambda^{-8} - \frac{s}{\mu}\lambda^{-3}} \tag{8}$$

Example

We can then plot this function for different values of s/μ to see how the voltage depends on the stretch.

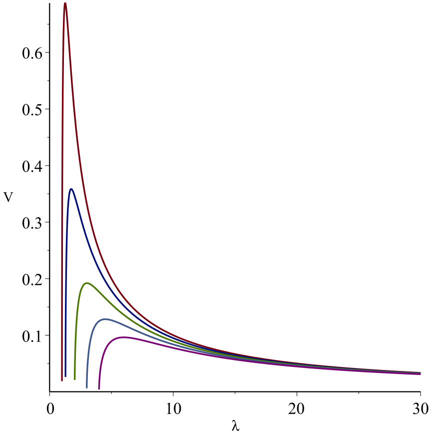

Figure: $\bar{V}(\lambda)$ for $s = 0, 1, 2, 3, 4$

References

- Dorfmann, L., Ogden, R.W., *Nonlinear Theory of Electroelastic and Magnetoelastic Interactions*, Springer, 2014.
- Zhao, X., Suo, Z., *Method to analyze electromechanical stability of dielectric elastomers*, Applied Physics Letters, 91, 2007.
- Norris, A., *Comment on "Method to analyze electromechanical stability of dielectric elastomers"*, Applied Physics Letters, 92, 2008.
- Huang, J., et al., *Maximizing the Energy Density of Dielectric Elastomer Generators Using Equi-Biaxial Loading*, Advanced Functional Materials, 23, 2013.